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UNIVERSITY OF ALBERTA

SOME OSCILLATION RESULTS FOR DIFFERENTIAL SYSTEMS

by

Lakshma Reddy Ganta

A THESIS SUBMITTED TO

THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

EDMONTON, ALBERTA

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a project entitled Some Oscillation Results for Differential Systems submitted by Lakshma Reddy Ganta in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

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To the loving memories of my parents, the late Sri. Yadava Reddy Ganta and the late Smt. Ramanamma Ganta.

ABSTRACT

In this thesis we consider second-order vector-matrix differential systems of the form

$$(P(t)y')' + Q(t)y = 0$$

$$(P(t)Y')' + Q(t)Y = 0$$

where $P(t)$ and $Q(t)$ are real continuous $n \times n$ symmetric matrix-valued functions with $P(t)$ being positive definite, $y(t)$ an n-dimensional vector and $Y(t)$ an $n \times n$ matrix function. By making use of the behaviour of the eigenvalues of the coefficient matrices, a number of oscillation results are proved. In addition, some comparison results for the system are also obtained.

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CHAPTER I

INTRODUCTION

The oscillation theory for the second order scalar differential equation

$$(p(t)x')' + q(t)x = 0 \tag{1}$$

where $p(t) > 0$, $q(t)$ are continuous real valued functions on the interval $[0, \infty)$, has been studied by many authors. The first fundamental result was the comparison theorem of Sturm. The Sturm comparison theorem in its restricted form was first discovered by Sturm [51] in 1836. Starting with Picone [50] in 1909, many researchers have since then improved Sturm's theorem to include a wide class of coefficients. For the formally self adjoint second order scalar differential equation

$$x'' + q(t)x = 0, \tag{2}$$

Kneser [30] proved in 1893 that if the limit

$$\lim_{t\to\infty} t^2 q(t) \tag{3}$$

exists and is equal to α then equation (2) is oscillatory if $\alpha > \frac{1}{4}$ and nonoscillatory if $\alpha < \frac{1}{4}$. In 1918 Fite [24] proved that $q(t) > 0$ together with

$$\int_0^\infty q(s)ds = +\infty \tag{4}$$

forms a sufficient condition for the oscillation of (2). Hille [27] generalized Kneser's results to obtain the following:

(i) equation (2) is oscillatory if

$$\liminf_{t\to\infty} t\int_t^\infty q(s)ds > \frac{1}{4}, \tag{5}$$

(ii) equation (2) is nonoscillatory if

$$\limsup_{t\to\infty} t\int_t^\infty p(s)ds < \frac{1}{4}. \tag{6}$$

It was in 1949 that Wintner [57] removed the condition $q(t) > 0$ in Fite's result. He proved that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T\int_0^t q(s)dsdt = +\infty \tag{7}$$

implies the oscillation of (2). Later Hartman [26] proved that in the case of nonexistence of the limit in (7), the following condition

$$-\infty < \liminf_{T\to\infty} \frac{1}{T}\int_0^T\int_0^t q(s)dsdt < \limsup_{T\to\infty} \frac{1}{T}\int_0^T\int_0^t q(s)dsdt \tag{8}$$

is also sufficient for the oscillation of (2). These results were further generalized by Olech, Opial and Wazewski [49] and by Willett [56]. Several researchers [26, 52, 55, 57] have obtained a number of results concerning the oscillation and nonoscillation of equation (1). A well known and more practical result is that of Leighton [39], who proved that equation (1) is oscillatory if

$$\int_0^\infty \frac{ds}{p(s)} = \int_0^\infty q(s)ds = +\infty. \tag{9}$$

Many researchers were and still are interested in knowing the extent to which the oscillation and nonoscillation criteria of the scalar equation (1) have analogues for the vector-matrix differential system

$$(P(t)Y')' + Q(t)Y = 0 \tag{10}$$

where $P(t)$, $Q(t)$ are real continuous n by n symmetric matrix-valued functions on $[0,\infty)$ with $P(t)$ positive definite for all $t \geq 0$. As in the scalar case, particular attention was paid to the case when $P(t) = I$, namely, the formally self adjoint differential system

$$Y'' + Q(t)Y = 0. \tag{11}$$

Etgen [18] has shown that (11) is oscillatory if

$$\lim_{t\to\infty} \int_0^t Tr\ Q(s)ds = \infty. \tag{12}$$

This result was extended by Noussair and Swanson [48], which in turn was further generalized by Allegretto and Erbe [1]. Etgen and Pawlowski [20] have shown that equation (11) is oscillatory if there exists a positive linear functional g such that

$$\lim_{t\to\infty} g\Big[\int_0^t Q(s)ds\Big]' = +\infty. \tag{13}$$

This result was generalized to the self adjoint equation (9). Etgen and Lewis [19] have shown that equation (9) is oscillatory if there exists a positive linear functional g such that the scalar equation

$$(g[P(t)]x')' + g[Q(t)]x = 0 \tag{14}$$

is oscillatory. It has been conjectured [28] that equation (11) is oscillatory at infinity whenever

$$\lim_{t\to\infty} \lambda_1\Big(\int_0^t Q(s)ds\Big) = +\infty$$

where for any symmetric matrix A, $\lambda_1(A)$ denotes the largest eigenvalue of A. In the case of $Q(t)$ positive semi-definite, the conjecture follows from condition (13). Similarly, the conjecture was first established under various additional conditions by Mingarelli [44, 45], Kwong *et al.* [38], Atkinson *et al.* [2], Kaper and Kwong [36], Butler and Erbe [6, 7], before it was finally settled by Byers, Harris and Kwong [10] and also by Kaper and Kwong [37]. It is clear that the eigenvalues of the coefficient matrices $P(t)$ and $Q(t)$ play a central role in obtaining oscillation results for equation (10). By drawing more from the intermediate eigenvalues, using a kind of "trade off", one can obtain a number of results for both equations (10) and (11).

In Chapter 2 we introduce notation, give some definitions and enumerate some basic results which are used in the subsequent chapters. In Chapter 3 we obtain oscillation criteria for the formally self adjoint system (11). Chapter 4 deals with oscillation criteria for the self adjoint equation (10), which can be regarded as a generalization of Leighton's result. In Chapter 5 we consider the problem of multiplying the coefficient matrices by matrices which preserve the oscillation property of the original systems.

CHAPTER II

PRELIMINARIES

2.1. Notation

The standard notations of differential and integral calculus are used throughout. We recall the following definitions [49] which we shall use in the coming chapters.

For any subset E of the real line R, $mes(E)$ denotes the Lebesgue measure of the subset E.

DEFINITION 2.1.1: Let $F(t)$ denote a continuous real valued function and ℓ be such that $-\infty \leq \ell \leq +\infty$. Then

$$\lim_{t \to \infty} \text{approx inf } F(t) = \ell$$

if

$$mes\{t : F(t) \leq \ell_1\} < +\infty \quad \text{for all} \quad \ell_1 < \ell$$

and

$$mes\{t : F(t) \leq \ell_2\} = +\infty \quad \text{for all } \ell_2 > \ell.$$

DEFINITION 2.1.2: Let $F(t)$ denote a continuous real valued function and L be a scalar such that $-\infty \leq L \leq +\infty$. Then

$$\lim_{t \to \infty} \text{approx sup } F(t) = L$$

if

$$mes\{t : F(t) \geq L_1\} = +\infty \quad \text{for all } L_1 < L$$

and

$$mes\{t : F(t) \geq L_2\} < +\infty \quad \text{for all } L_2 > L.$$

DEFINITION 2.1.3: For a continuous real valued function $F(t)$ and a scalar m we write

$$\lim_{t\to\infty} \text{approx } F(t) = m$$

if

$$\lim_{t\to\infty} \text{approx inf } F(t) = \lim \text{approx } \sup_{t\to\infty} F(t) = m.$$

From these definitions, we have, for a continuous real valued function $F(t)$,

$$\liminf_{t\to\infty} F(t) \leq \lim_{t\to\infty} \text{approx inf } F(t) \leq \lim_{t\to\infty} \text{approx sup } F(t) \leq \limsup_{t\to\infty} F(t).$$

2.2. Matrix theory

Unless stated otherwise capital letters will be used to denote real square matrices of order $n \times n$. The transpose of a matrix A is denoted by the matrix A^*. The symbol I will denote the identity matrix. The dimension of I will be clear from the context. The symbol O will denote the zero element. Again, it would be evident from the context whether it is the zero matrix, zero vector or a scalar. For n-tuples in R^n and scalars, lower case letters and Greek letters

are used respectively, and will be mentioned at the beginning of their usage from time to time if there is ambiguity.

Standard notations of matrix theory are also used throughout. Let A be an $n \times n$ matrix. Then A^{-1} will denote the inverse of A if A is nonsingular, i.e., $\det A \neq 0$. A matrix valued function $A(t) = (a_{ij}(t))$ is said to be continuous if each of its entries $a_{ij}(t)$ is continuous. Similarly $A(t)$ is said to have a property from calculus if and only if its entries $a_{ij}(t)$ all have that property. For obvious reasons, symmetric matrices will play an important role. We denote the set of all symmetric matrices of order $n \times n$ by S. With the usual addition of matrices and multiplication of a matrix with a scalar, S is a linear vector space over the reals. For any real symmetric matrix A, its eigenvalues, all of which are necessarily real, will be denoted by $\lambda_k(A)$, $1 \leq k \leq n$, and are assumed to be ordered so that

$$\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A).$$

As usual $Tr\ A$ will denote the trace of the matrix A. We have

$$Tr\ A = \sum_{i=1}^{n} \lambda_i(A).$$

A symmetric matrix $A \in S$ is *positive semi-definite* (*positive definite*) if for all $x \in R^n$, $x \neq 0$, the usual inner product (x, Ax) is non-negative (positive). To say A is positive definite (positive semi-definite) the notation $A > 0$ ($A \geq 0$) will be used throughout. For A and B in S, $A > B$ means $A - B > 0$.

A linear function $\phi : S \to R$ satisfying

$$\phi(A+B) = \phi(A) + \phi(B) \quad \text{and}$$

$$\phi(\lambda A) = \lambda\phi(A)$$

where $\alpha \in R$, $A, B \in S$, is called *positive* and *normalized* provided

$$\phi(A) > 0 \quad \text{if} \quad A > 0 \text{ and}$$

$$\phi(I) = 1.$$

A non-linear functional $q : S \to R$ is called *superadditive* and (*positively*) *superhomogeneous* if

$$q(A+B) \geq q(A) + q(B) \quad \text{and}$$

$$q(\alpha A) \geq \alpha q(A) \quad \text{for} \quad \alpha \geq 0$$

where $A, B \in S$, and is called *positive* and *normalized* if

$$q(A) > 0 \quad \text{if} \quad A > 0 \text{ and}$$

$$q(I) = 1.$$

Correspondingly, a functional $p : S \to R$ is called *subadditive* and (*positively*) *subhomogeneous* if

$$p(A+B) \leq p(A) + p(B) \quad \text{and}$$

$$p(\alpha A) \leq \alpha p(A) \quad \text{for } \alpha \geq 0.$$

Note that the conditions on q imply that q is concave and hence is continuous on S. A similar remark is valid for p.

As a positive linear functional ϕ is continuous, we have for integrable $A(t)$,

$$\phi\Big(\int_a^t A(s)ds\Big) = \int_a^t \phi[A(s)]ds, \quad a,t \in R^+$$

and

$$\phi[B'(t)] = (\phi[B(t)])', \quad t \in R^+$$

whenever $B(t)$ is differentiable.

2.3. Results on symmetric matrices

The purpose of this section is to collect some basic results based on the topics discussed in the previous section. These results will be used repeatedly in the subsequent chapters. Discussion of undefined terms and detailed accounts of the topics discussed here and in the previous section can be found in the references [5, 41, 42]. In the following discussion, all matrices will be assumed symmetric.

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

be a symmetric matrix. The principal sub-matrices of A are

$$A_1 = [a_{11}], \quad A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \dots A_k = \begin{bmatrix} a_{11} & \dots & a_{1k} \\ \vdots & & \\ a_{k1} & \dots & a_{kk} \end{bmatrix}$$

These matrices are all symmetric and we have

$$\lambda_{j+1}(A_{k+1}) \leq \lambda_j(A_k) \leq \lambda_j(A_{k+1})$$

where $\lambda_j(A_k)$ denotes the j^{th} characteristic root of the principal submatrix

$$A_k = \begin{bmatrix} a_{11} & \dots & a_{1k} \\ \vdots & \vdots & \vdots \\ a_{k1} & \dots & a_{kk} \end{bmatrix}.$$

LEMMA 2.3.1. *If $A \geq 0$ then all eigenvalues of A are non-negative.*

LEMMA 2.3.2. *If $A \geq B$, then $Tr\ A \geq Tr\ B$.*

LEMMA 2.3.3. *If $A \geq B$ and $C \geq D$, then $A + C \geq B + D$.*

LEMMA 2.3.4. *(i) If $0 \leq A$ and $A \not\leq I$, then $Tr\ A > 1$.*

(ii) If $A \not\leq B$ and $B \geq C$, then $A \not\leq C$.

LEMMA 2.3.5. *If $A > B$ and $C = C^*$, then $CAC > CBC$.*

LEMMA 2.3.6. *(i) If $A(t) \geq 0$ for $t \in (a, b)$, then*

$$\int_a^b A(t)dt \geq 0.$$

(ii) If $A(0) \geq 0$ and $A'(t) \geq 0$, then $A(t) \geq 0$.

(iii) $\int_a^b Tr\ \{A(t)^{-1}A'(t)A(t)^{-1}\}dt = Tr\ \{A(a)^{-1}\} - Tr\ \{A(b)^{-1}\}$.

LEMMA 2.3.7. *If A and B are positive semi-definite and $A^2 \geq B^2$, then $A \geq B$.*

LEMMA 2.3.8. *Let C and A be positive definite. If $ACA \geq BCB$, then $A \geq B$.*

For $1 \leq k \leq n$, and a symmetric matrix A, we define

$$\sigma_k(A) = \lambda_1(A) + \cdots + \lambda_k(A)$$

and

$$\rho_k(A) = \lambda_n(A) + \lambda_{n-1}(A) + \cdots + \lambda_{n-k+1}(A)$$

where the n eigenvalues of A are ordered in descending order of magnitude.

LEMMA 2.3.9. *If A and B are symmetric, then*

(i) $\sigma_k(A+B) \leq \sigma_k(A) + \sigma_k(B)$;

(ii) $\rho_k(A) + \rho_k(B) \leq \rho_k(A+B)$;

(iii) $|\sigma_k(A)|^2 \leq k\sigma_k(A^2)$;

(iv) $|\rho_k(A)|^2 = |\sigma_k(-A)|^2 \leq k\sigma_k(A^2)$;

(v) $Tr\ (A) \leq (n-k+1)\sigma_k(A) \leq n\sigma_k(A)$.

2.4. Definitions

There is a vast literature on the subject of the existence and location of zeros of solutions of differential equations. In the absence of formal expressions for the solution of a differential equation, attention is directed to the problem of obtaining essential characteristics of the solutions of the equation, by directly analyzing the equation itself. One such qualitative study is the subject of oscillation theory.

Consider the second-order scalar differential equation

$$\left(p(t)x'\right)' + q(t)x = 0, \tag{2.4.1}$$

where $p(t) > 0$, $q(t)$ are continuous, real valued functions on the interval $R^+ = [0, \infty)$ and $'$ denotes differentiation with respect to t. The oscillation theory for the equation (2.4.1) has been studied by many authors [25, 51, 55, 57] and continues to flourish.

DEFINITION 2.4.1: Equation (2.4.1) is said to be *oscillatory* on the interval $R^+ = [0, \infty)$ if one (hence all) solution $x(t) \not\equiv 0$ of (2.4.1) has arbitrarily large zeros. The equation is said to be *nonoscillatory* if it is not oscillatory. Oscillatory on the interval $R^+ = [0, \infty)$ and oscillatory at ∞ mean the same.

The beginning of such a qualitative study can be traced back to the classic paper of Sturm [51], wherein his celebrated comparison theorem was proved.

We are mainly concerned with the self adjoint differential system

$$(P(t)Y')' + Q(t)Y = 0 \tag{2.4.2}$$

where $P(t), Q(t), Y(t)$ are all $n \times n$ real continuous matrix functions. We also assume that $P(t)$, $Q(t)$ are symmetric and that $P(t)$ is positive definite for $t \in [0, \infty)$. Also, particular attention will be paid to the special case when $P(t) = I$, that is, to the equation

$$Y'' + Q(t)Y = 0. \tag{2.4.3}$$

We also have the vector systems associated with equations (2.4.2) and (2.4.3)

$$(P(t)y')' + Q(t)y = 0 \tag{2.4.4}$$

and

$$y'' + Q(t)y = 0 \tag{2.4.5}$$

where $y = \text{col}(y_1, y_2, \ldots, y_n)$ is an n-vector.

DEFINITION 2.4.2: Two distinct points $t_0,\ t_1 \in J = [a, b]$ are said to be (*mutually*) *conjugate* relative to equation (2.4.4) if there exists a nontrivial solution $y(t)$ of (2.4.4) such that $y(t_0) = y(t_1) = 0$.

DEFINITION 2.4.3: Equation (2.4.4) is said to be *disconjugate* on an interval J if every nontrivial solution of equation (2.4.4) vanishes at most once in J.

DEFINITION 2.4.4: Equation (2.4.4) is said to be *oscillatory* on $J = [a, \infty),\ a \geq 0$, if for each $t_0 > a$ there exists a $t_1 > t_0$ such that equation (2.4.4) is not disconjugate on $[t_1, \infty)$.

DEFINITION 2.4.5: The system (2.4.5) is said to be *right-disfocal* on J if, for every pair of points $a,\ b \in J,\ a < b$, the only solution y satisfying the condition $y(a) = y'(b) = 0$ is the trivial solution.

DEFINITION 2.4.6: The system (2.4.5) is said to be *left-disfocal* on J if, for every pair of points $a,\ b \in J,\ a < b$, the only solution y satisfying the condition $y'(a) = y(b) = 0$ is the trivial solution.

For the first order system

$$y' + Q(t)y = 0 \tag{2.4.6}$$

we have the following concepts [47].

DEFINITION 2.4.7: A nontrivial solution $y = \text{col}(y_1, y_2, \ldots, y_n)$ of (2.4.6) is said to be *oscillatory* on an interval J if $y_k(t_k) = 0$ for some $t_k \in J$, $k = 1, 2, \ldots, n$.

DEFINITION 2.4.8: The first order system (2.4.6) is said to be *oscillatory* if it has at least one oscillatory solution vector.

DEFINITION 2.4.9: The first order system (2.4.6) is said to be *suborthogonal* on J if, for any nontrivial solution vector y and for any pair of points a, $b \in J$, the inner product $(y(a), y(b)) = \sum_{k=1}^{n} y_k(a) y_k(b) > 0$.

DEFINITION 2.4.10: A solution $Y(t)$ of the matrix differential equation (2.4.2) is called *nontrivial* if $\det Y(t) \neq 0$ for at least one $t \in [0, \infty)$.

DEFINITION 2.4.11: A solution $Y(t)$ of the matrix differential equation (2.4.2) is said to be *non-singular* at $a \in [0, \infty)$ if $\det Y(a) \neq 0$.

DEFINITION 2.4.12: A nontrivial solution $Y(t)$ of the matrix differential equation (2.4.2) is said to be *oscillatory* at ∞ if $\det Y(t)$ has infinitely many zeros in $[0, \infty)$. If it is not oscillatory then it is called *non-oscillatory*.

If one hopes to develop an analogue of the classical theory of oscillation for the scalar case, namely equation (2.4.1), for the linear system (2.4.2), one needs to consider only certain classes of solutions of the equation of type (2.4.2). The following example [48] clarifies the situation. We know that the linear scalar equation

$$x'' + x = 0 \tag{2.4.7}$$

has only solutions of the form

$$x(t) = c_1 \sin t + c_2 \cos t \tag{2.4.8}$$

where c_1, c_2 are arbitrary constants. As all these solutions are oscillatory, equation (2.4.7) is oscillatory. Now consider the matrix analogue of the above equation, namely the matrix differential equation

$$Y'' + Y(t) = 0. \tag{2.4.9}$$

As in the case of equation (2.4.7), one expects to see that equation (2.4.9) is oscillatory (shortly we will give the precise definition). But the following matrix

$$Y(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

is a solution of the matrix differential equation (2.4.9). As $\det Y(t) = 1$, we see that $Y(t)$ is a non-oscillatory solution of (2.4.9). Hence one has to consider a subclass of the class of solutions of equations of type (2.4.2). This subclass is that of all *prepared solutions*.

First, note that if $Y = Y(t)$ is a solution of the equation (2.4.2), then

$$Y^*(t)[P(t)Y'(t)] - [P(t)Y'(t)]^*Y(t) \equiv C \tag{2.4.10}$$

where C is an $n \times n$ constant matrix.

DEFINITION 2.4.13: A non-trivial solution $Y(t)$ of the matrix differential equation (2.4.2) will be called *prepared* (or *conjoined*, or *self-conjugate*) if

$$Y(t)^*[P(t)Y'(t)] - [P(t)Y'(t)]^*Y \equiv 0. \tag{2.4.11}$$

Note that a non-trivial solution $Y(t)$ of (2.4.2) is prepared if and only if $[P(t)Y'(t)]^*Y(t)$ is symmetric.

As a consequence of (2.4.10) and (2.4.11), $Y(t)$ is a prepared solution of (2.4.2) if and only if there exists $a \in [0, \infty)$ such that

$$Y^*(a)[P(a)Y'(a)] - [P(a)Y'(a)]^*Y(a) = 0.$$

Thus, prepared solutions can be obtained by starting out with appropriate initial values.

DEFINITION 2.4.14: The matrix differential equation (2.4.2) is said to be *oscillatory* if the determinant of every nontrivial prepared solution vanishes on $[a, \infty)$ for each $a > 0$.

Oscillation of the matrix differential equation (2.4.2) is equivalent to the oscillation of the vector differential equation (2.4.4) since any solution of (2.4.4) is of the form $y(t) = Y(t)\alpha$ for some constant vector α and some nontrivial prepared solution $Y(t)$ of (2.4.2). For this reason, very often, one studies equation (2.4.2) to study the oscillatory properties of equation (2.4.4).

Two basic techniques used in studying oscillation and non-oscillation of the scalar equation (2.4.1) have analogous approaches available for the differential systems. These techniques are generally known as the nonlinear Riccati integral equation approach and the variational approach [26].

2.5. Riccati and Variational Methods

In this section we briefly recall the essentials of the aforementioned approaches in connection with the study of oscillation and nonoscillation of differential systems.

Riccati Method: Let $Y = Y(t)$ be any solution (not necessarily prepared) of the matrix differential equation

$$(P(t)Y'(t))' + Q(t)Y(t) = 0. \tag{2.5.1}$$

Taking $R(t) = -P(t)Y'(t)Y^{-1}(t)$ and using the fact that $Y(t)$ is a solution of (2.5.1), we get the Riccati equation

$$R'(t) = Q(t) + R(t)P^{-1}(t)R(t). \tag{2.5.2}$$

Equation (2.5.2) gives the Riccati integral equation

$$R(t) = R(a) + \int_0^t Q(s)ds + \int_a^t R(s)P^{-1}(s)R(s)ds. \tag{2.5.3}$$

If $R(t)$ is any solution of (2.5.2), from (2.5.2) it follows that $R(t)^*$ is also a solution of (2.5.2). From this observation and by the uniqueness theorem it follows that $R(t)$ is symmetric for all t if $R(t)$ is symmetric for some value of t.

If $Y = Y(t)$ is a nontrivial prepared solution of the matrix differential equation (2.5.1) with $\det Y(t) \neq 0$ for $t > a$, then $R(t) = -P(t)Y'(t)Y^{-1}(t)$ is a symmetric solution of (2.5.2) on the infinite interval $[a, \infty)$. Thus the matrix differential equation (2.5.1) is oscillatory if and only if the Riccati integral equation (2.5.3) has no continuous solution in some neighborhood of infinity.

Variational Method: As in the case of scalar equations, variational principles can be employed to obtain oscillation criteria for the matrix differential system [26].

For any subinterval $[a,b]$ of $[0,\infty)$, define

$$A_1(a,b) = \{y : [a,b] \to R^n,\ y(a) = y(b) = 0,\ y \in AC[a,b] \text{ and } y' \in L^2(a,b)\},$$

as the admissible class of vector functions.

We recall the following oscillation criterion for the matrix differential equation

$$Y'' + Q(t)Y = 0. \tag{2.5.4}$$

THEOREM 2.5.1 [26]. *The matrix differential equation (2.5.4) is oscillatory if and only if there is a sequence of intervals* $[a_n, b_n]$, *with* $\lim_{n\to\infty} a_n = \infty$, *and a sequence of functions* $Q_n \in A_1(a_n, b_n)$, *such that*

$$\int_{a_n}^{b_n} \{|Q_n'(t)|^2 - Q_n^*(t)Q(t)Q_n(t)\}dt < 0.$$

CHAPTER III

3.1. Introduction

In this chapter we are concerned with the second order $n \times n$ matrix differential equation

$$Y'' + Q(t)Y = 0, \quad t \in [a, \infty) \tag{3.1.1}$$

where $Y(t)$, $Q(t)$ are $n \times n$ real continuous matrix functions and $Q(t)$ is symmetric. We know that the corresponding scalar equation

$$y'' + q(t)y = 0, \quad t \in [a, \infty) \tag{3.1.2}$$

is nonoscillatory on $[a, \infty)$ if and only if the Riccati integral equation

$$r(t) = r(t_0) + \int_{t_0}^{t} q(s)ds + \int_{t_0}^{t} r^2(s)ds \tag{3.1.3}$$

has a continuous solution on $[t_0, \infty)$ for some t_0. If $y = y(t)$ with $y(t) \neq 0$ for $t \geq t_0$ is a solution of the scalar equation (3.1.2) then $r(t) = -y'/y$ is a solution of the associated Riccati integral equation (3.1.3). Similarly, we have seen that if $Y = Y(t)$ is a nontrivial prepared solution of the second order matrix differential equation (3.1.1), then $R(t) = -Y'Y^{-1}$ is a symmetric solution of the corresponding matrix Riccati equation

$$R(t) = R(t_0) + Q_1(t) + F(t) \tag{3.1.4}$$

where

$$Q_1(t) = \int_{t_0}^{t} Q(s)ds \tag{3.1.5}$$

$$F(t) = \int_{t_0}^{t} R^2(s). \tag{3.1.6}$$

For the scalar equation (3.1.2) Hartman [25] has shown that if equation (3.1.2) is nonoscillatory on $[a,\infty)$, then a necessary and sufficient condition that

$$\int^{\infty} \left(\frac{y'}{y}\right)^2 dt < \infty$$

holds for a solution $y \neq 0$ is that

$$\liminf_{T\to\infty} \frac{1}{T}\int_a^T \int_a^t q(s)dsdt > -\infty.$$

Hence when studying the oscillation behaviour of the scalar equation (3.1.2), one has to consider the two distinct cases namely, (i) the above limit does hold, and (ii) the above limit does not hold.

3.2. Recent results

From the above considerations it follows that one has to consider the extended real-valued function L which is defined on the class of $n\times n$ continuous real symmetric matrices defined on $[a,\infty)$ by

$$L(Q) \equiv \liminf_{T\to\infty} \frac{1}{T}\int_a^T \int_a^t Tr\ Q(s)dsdt.$$

The value of $L(Q)$ will play a central role in the discussion to follow. The following lemma is due to Mingarelli [44] [see also Butler, Erbe and Mingarelli [9]], and is the systems analogue of the scalar result of Hartman [25] mentioned earlier in this chapter.

LEMMA 3.2.1 [9]. *Assume that the second order matrix differential equation (3.1.1) is nonoscillatory on* $[a,\infty)$. *Then a necessary and sufficient condition that*

$$\lim_{T\to\infty}\int_t^T R^2(s)ds$$

exists for any solution $R=-Y'Y^{-1}$ *of*

$$R(t)=R(t_0)+\int_{t_0}^t Q(s)ds+\int_{t_0}^t R^2(s)ds$$

where $Y(t)$ *is a prepared solution of equation (3.1.1) is that*

$$L(Q)=\liminf_{T\to\infty}\frac{1}{T}\int_a^T\int_a^t Tr\,Q(s)dsdt>-\infty.$$

□

As in the scalar case, by considering the two distinct cases $L(Q)=-\infty$, $L(Q)>-\infty$ separately, Butler, Erbe and Mingarelli [9] obtained a number of oscillation criteria for the equation (3.1.1), some of which are generalizations of the scalar results mentioned earlier.

THEOREM 3.2.2 [9]. *Assume* $L(Q)>-\infty$. *Then equation (3.1.1) is oscillatory if any of the following conditions hold.*

(A) $\limsup_{T\to\infty}\frac{1}{T}\int_a^T\lambda_1\Big(\int_a^t Q(s)ds\Big)dt=+\infty;$

(B) $\limsup_{T\to\infty}\frac{1}{T}\int_a^T\Big[\lambda_1\Big(\int_a^t Q(s)ds\Big)\Big]^2dt=+\infty;$

(C) $\lim\operatorname{approx}\sup_{T\to\infty}\lambda_1\Big(\int_a^T Q(s)ds\Big)=+\infty;$

(D) $\limsup\limits_{T\to\infty}$ approx inf $\lambda_1\Big(\int_a^T Q(s)ds\Big) = -\infty.$

□

THEOREM 3.2.3 [9]. *Assume $L(Q) = -\infty$. Then (3.1.1) is oscillatory if*

$$\limsup_{T\to\infty} \text{ approx sup } \lambda_n\Big(\int_a^T Q(s)ds\Big) > -\infty. \qquad \square$$

As mentioned earlier, Wintner [57] showed that the scalar equation (3.1.2) is oscillatory if

$$\lim_{T\to\infty} \frac{1}{T}\int_a^T \int_a^t q(s)dsdt = +\infty.$$

A partial generalization of this result to systems is

THEOREM 3.2.4 [9]. *Assume that*

$$\lambda_1\Big(\int_a^t Q(s)ds\Big) > 0$$

for sufficiently large t, and that

$$\lim_{t\to\infty} \left|\frac{\lambda_1\Big(\int_a^t Q(s)ds\Big)}{\lambda_n\Big(\int_a^t Q(s)ds\Big)}\right| > 0.$$

Then (3.1.1) is oscillatory if

$$\lim_{T\to\infty} \frac{1}{T}\int_a^T \lambda_1\Big(\int_a^t Q(s)ds\Big)dt = +\infty.$$

□

Oscillation results for the equation (3.1.1), which involve the quotient of $\lambda_1\Big(\int_a^t Q(s)ds\Big)$ and $\lambda_n\Big(\int_a^t Q(s)ds\Big)$ with some other conditions, were also given by Butler and Erbe [6].

3.3. Oscillation theorems

Consider the second order matrix differential equation

$$Y'' + Q(t)Y = 0, \quad t \in [a, \infty). \tag{3.3.1}$$

Once again, by considering two different cases depending on the value of

$$L(Q) = \liminf_{T\to\infty} \frac{1}{T}\int_a^T \int_a^t Tr\ Q(s)dsdt,$$

and by using the behavior of the intermediate eigenvalues one can obtain the following results.

THEOREM 3.3.1. *The matrix differential equation (3.3.1) is oscillatory if*

(i) $L(Q) > -\infty,$ *(3.3.2)*

(ii) $\liminf\limits_{t\to\infty}\ \sigma_k\Big(\int_a^t Q(s)ds\Big) = -\infty,$ *(3.3.3)*

for some $1 \le k \le n$, *where* $\sigma_k(A) = \sum\limits_{i=1}^{k} \lambda_i(A)$.

THEOREM 3.3.2. *The matrix differential equation (3.3.1) is oscillatory if*

(i) $L(Q) = -\infty,$

(ii) $\limsup\limits_{t_0\to\infty}\ \rho_k\Big(\int_a^t Q(s)ds\Big) > -\infty,$ *(3.3.4)*

for some $1 \le k \le n$, *where* $\rho_k(A) = \sum_{i=1}^{k} \lambda_{n-k+i}(A)$.

Before we give the proofs of these theorems, we note that by taking $Q(t) = \operatorname{diag}(q_1(t), \ldots, q_n(t))$, with suitable $q_i(t)$, one can construct examples to illustrate that any one of the (ii) conditions may hold for some k_0, $1 < k_0 \le n$, but may fail for all $1 \le k < k_0$. Thus each of the above results gives a sequence of n tests for the oscillation of equation (3.3.1).

PROOF OF THEOREM 3.3.1: Assume that equation (3.3.1) is nonoscillatory on $[a, \infty)$. Let $Y = Y(t)$ be a nontrivial prepared solution of (3.3.1) with $\det Y(t) \neq 0$ for $t \ge a$. Then $Z = -Y'Y^{-1}$ is a symmetric solution of the Riccati equation

$$Z(t) = Z(a) + \int_a^t Q(s)ds + \int_a^t Z^2(s)ds \tag{3.3.5}$$

for $t \ge a$. From the assumption made at the beginning of this proof and condition (3.3.2), we conclude, by using Lemma [3.2.1], that

$$\lim_{T \to \infty} \int_t^T Tr\ Z^2(s)ds \tag{3.3.6}$$

exists as a finite number. From (3.3.3), it follows that for $M > 0$,

$$mes\Big\{t : \sigma_k\Big(\int_a^t Q(s)ds\Big) < -M\Big\} = +\infty.$$

From (3.3.6) and (3.3.5) we get

$$Z(t) + \int_t^\infty Z^2(s)ds + C = \int_{t_0}^t Q(s)ds$$

where $C = -Z(a) - \int_a^\infty Z^2(s)ds$. Since $Z^2(s) \geq 0$, we have

$$\sigma_k(Z(t)) \leq \sigma_k\Big(Z(t) + \int_t^\infty Z^2(s)ds\Big)$$

$$= \sigma_k\Big(\int_a^t Q(s)ds - C\Big).$$

The subadditive nature of σ_k gives

$$\sigma_k(Z(t)) \leq \sigma_k\Big(\int_a^t Q(s)ds\Big) + \sigma_k(-C).$$

If $M > 1 + |\sigma_k(-C)|$, then

$$\sigma_k\Big(\int_a^t Q(s)ds\Big) < -M$$

which implies $\sigma_k(Z(t)) < -1$. Hence

$$mes\{t : \sigma_k(Z(t)) < -1\} = +\infty.$$

This in turn implies

$$\int_a^\infty \big(\sigma_k Z(t)\big)^2 dt = +\infty.$$

Now using the fact that for a symmetric matrix A we have $|\sigma_k(A)|^2 \leq k\sigma_k(A^2)$, we arrive at

$$\int_a^\infty k\sigma_k(Z^2(t))dt = +\infty$$

and whence

$$\int_a^\infty Tr\ Z^2(t)dt = +\infty,$$

which is a contradiction to (3.3.6). Hence equation (3.3.1) is oscillatory.

PROOF OF THEOREM 3.3.2: Assume $L(Q) = -\infty$ and (3.3.4) holds. If equation (3.3.1) is nonoscillatory, then we may assume that there exists a solution $Y = Y(t)$ of (3.3.1), with $\det Y(t) \neq 0$ on $[a, \infty)$. Then $Z = -Y'Y^{-1}$ is a symmetric solution of the Riccati equation

$$Z(t) = Z(a) + \int_a^t Z^2(s)ds + \int_a^t Q(s)ds. \tag{3.3.5}$$

Since

$$\sigma_k\Big(-\int_a^t Q(s)ds\Big) = -\rho_k\Big(\int_a^t Q(s)ds\Big),$$

we have

$$\begin{aligned}\sigma_k\Big(Z(t) - \int_a^t Q(s)ds\Big) &\leq \sigma_k(Z(t)) + \sigma_k\Big(-\int_a^t Q(s)ds\Big) \\ &= \sigma_k(Z(t)) - \rho_k\Big(\int_a^t Q(s)ds\Big).\end{aligned}$$

Upon noting that for symmetric matrices A we have $Tr(A) \leq (n-k+1)\sigma_k(A) \leq n\sigma_k(A)$, we have

$$\begin{aligned}\frac{1}{n}\ Tr\ \int_a^t Z^2(s)ds &= \frac{1}{n}\ Tr\Big(Z(t) - \int_a^t Q(s)ds - Z(a)\Big) \\ &\leq \sigma_k\Big(Z(t) - Z(a) - \int_a^t Q(s)ds\Big) \\ &\leq \sigma_k(Z(t)) - \rho_k(Z(a)) - \rho_k\Big(\int_a^t Q(s)ds\Big).\end{aligned}$$

That is,

$$\rho_k(Z(a)) + \frac{1}{n}\ Tr\Big(\int_a^t Z^2(s)ds\Big) \leq \sigma_k(Z(t)) - \rho_k\Big(\int_a^t Q(s)ds\Big).$$

As

$$\frac{1}{n}\ Tr\ \int_a^t Z^2(s)ds \geq \frac{1}{n}\int_a^t \sigma_k(Z^2(s))ds,$$

we obtain

$$\rho_k(Z(a)) + \frac{1}{n}\ \int_a^t \sigma_k(Z^2(s))ds \leq \sigma_k(Z(t)) - \rho_k\Big(\int_a^t Q(s)ds\Big). \qquad (3.3.7)$$

If

$$m = \lim_{t\to\infty} \operatorname{approx\,sup}\, \rho_k\Big(\int_a^t Q(s)ds\Big),$$

then for any $\epsilon > 0$, it follows that

$$mes\Big\{t : \rho_k\Big(\int_a^t Q(s)ds\Big) \geq m - \epsilon\Big\} = +\infty.$$

We observe that if

$$\rho_k\Big(\int_a^t Q(s)ds\Big) \geq m - \epsilon,$$

then

$$\rho_k(Z(a)) + \frac{1}{n}\int_a^t \sigma_k(Z^2(s))ds \leq \sigma_k(Z(t)) - m + \epsilon$$

whence

$$mes\big\{t : \rho_k(Z(a)) + \frac{1}{n}\int_a^t \sigma_k(Z^2(s))ds \leq \sigma_k(Z(t)) - m + \epsilon\big\} = +\infty.$$

Since $L(Q) = -\infty$, it follows that

$$\int_{t_0}^t Tr\ Z^2(s)ds \to +\infty$$

as $t \to \infty$ and therefore

$$\sigma_k\Big(\int_a^t Z^2(s)ds\Big) \to +\infty$$

as $t \to +\infty$. We note that if the set E is defined by

$$E = \left\{ t \in [a+1, \infty) : \frac{1}{2n} \int_a^t \sigma_k(Z^2(s))ds \leq \sigma_k(Z(t)) \right\},$$

then $mes(E) = +\infty$. By defining $P(t)$ as

$$P(t) = \int_a^t \sigma_k(Z^2(s))ds,$$

we have

$$P'(t) + \sigma_k(Z^Z(t)) \geq \frac{1}{k}(\sigma_k(Z(t))^2$$

and for $t \in E$ we have

$$\sigma_k(Z(t)) \geq \frac{1}{2n} \int_a^t \sigma_k(Z^2(s))ds$$

which implies that

$$(\sigma_k(Z(t)))^2 \geq \frac{1}{4n^2} P^2(t).$$

That is,

$$P'(t) \geq \frac{1}{4n^2 k} P^2(t).$$

Integrating over the set E gives

$$\frac{1}{P(a+1)} \geq \int_E \frac{p'(t)}{p^2(t)} dt \geq \frac{1}{4n^2 k} \cdot \mu(E) = +\infty,$$

a contradiction. Hence equation (3.3.1) is oscillatory. □

A simple modification of the above proof leads to the following result.

THEOREM 3.3.3. *The differential system (3.3.1) is oscillatory if*

(i) $L(Q) = -\infty$;

(ii) $\limsup_{t\to\infty}^{\text{approx}} Tr\Big(\int_a^t Q(s)ds\Big) > -\infty.$ *(3.3.8)*

3.4. More oscillation criteria.

For the scalar equation

$$y'' + q(t)y = 0, \tag{3.4.1}$$

Olech, Opial and Wazewski [49] proved that

$$\lim_{T\to\infty}^{\text{approx}} \int_a^T q(s)ds = +\infty$$

or

$$\liminf_{T\to\infty}^{\text{approx}} \int_a^T q(s)ds < \limsup_{T\to\infty}^{\text{approx}} \int_a^T q(s)ds$$

implies oscillation of (3.4.1). As noted before, Wintner [57] established that equation (3.4.1) is oscillatory in the case

$$\lim_{T\to\infty} \frac{1}{T}\int_a^T \int_a^t q(s)dsdt = +\infty.$$

In case the above limit fails to exist, Hartman [26] showed that

$$-\infty < \liminf_{T\to\infty} \frac{1}{T}\int_a^T \int_a^t q(s)dsdt < \limsup_{T\to\infty} \frac{1}{T}\int_a^T \int_a^t q(s)dsdt$$

guarantees the oscillation of the equation (3.4.1). We now concern ourselves with the matrix equation

$$Y'' + Q(t)Y = 0. \tag{3.4.2}$$

Recently Erbe [15] generalized the above mentioned oscillation criterion for the scalar equation to derive an oscillation criterion for the second order differential system (3.4.2). For this he exploited the behaviour of the largest eigenvalue of the integral of the coefficient matrix $Q(t)$ together with the value of $L(Q)$. More precisely, Erbe proved

THEOREM 3.4.1 [15]. *The matrix differential equation (3.4.2) is oscillatory if*

(i) $L(Q) > -\infty$;

(ii) there exists $\delta > 0$ *such that for all large* t_0, *we have*

$$\eta(t_0) - \mu(t_0) \geq \delta$$

where

$$\eta(t_0) \equiv \limsup_{T\to\infty} \frac{1}{T}\int_{t_0}^{T} \lambda_1\Big(\int_{t_0}^{t} Q(s)dsdt\Big)$$

$$\mu(t_0) \equiv \liminf_{T\to\infty} \frac{1}{T}\int_{t_0}^{T} \lambda_1\Big(\int_{t_0}^{t} Q(s)dsdt\Big).$$

We are interested in determining the extent to which the criteria given by Olech, Opial and Wazewski mentioned earlier have analogues for the matrix equation. In this direction, we have

THEOREM 3.4.2. *The matrix differential equation (3.4.2) is oscillatory if*

(i) $L(Q) > -\infty$;

(ii) there exists $\delta > 0$ *and* k, $1 \leq k \leq n$, *such that for all large* t_0,

$$\eta_k(t_0) - \mu_k(t_0) \geq \delta \tag{3.4.3}$$

where

$$\eta_k(t_0) = \limsup_{T\to\infty} \frac{1}{T}\int_{t_0}^{T} \sigma_k\Big(\int_{t_0}^{t} Q(s)ds\Big)dt,$$

$$\mu_k(t_0) = \liminf_{T\to\infty} \frac{1}{T}\int_{t_0}^{T} \sigma_k\Big(\int_{t_0}^{t} Q(s)ds\Big)dt.$$

PROOF: Assume that the equation (3.4.2) is nonoscillatory. Let $Y = Y(t)$ be a nontrivial prepared solution of (3.4.2) with $\det Y(t) \neq 0$ for $t \geq t_0$. Then $Z = -Y'Y^{-1}$ is a solution of the Riccati equation

$$Z(t) = Z(t_0) + \int_{t_0}^{t} Q(s)ds + \int_{t_0}^{t} Z^2(s)ds$$

for $t \geq t_0$. First we suppose that both $\eta_k(t_0)$ and $\mu_k(t_0)$ are finite. By defining

$$Q_1(t;t_0) \equiv \int_{t_0}^{t} Q(s)ds \tag{3.4.4}$$

$$A(t;t_0) \equiv Z(t_0) + \int_{t_0}^{t} Z^2(s)ds \tag{3.4.5}$$

and using Lemma [3.2.1], we see that $\lim_{t\to\infty} A(t;t_0)$ exists and hence $\lim_{t\to\infty} Tr\ A(t;t_0)$ exists. Take $\epsilon > 0$ such that $\epsilon < \frac{\delta}{3}$. Using the fact that

$$\lim_{T\to\infty} \int_{t}^{T} Tr\ Z^2(s)ds \tag{3.4.6}$$

exists as a finite number, we may suppose that t_0 is sufficiently large so that

$$\int_{s}^{t} Tr\ Z^2(u)du < \epsilon \tag{3.4.7}$$

for $t_0 \leq s < t < +\infty$ and also $\sigma_k(Z^2(t_0)) < \frac{\epsilon^2}{k}$. The Riccati equation takes the form

$$Z(t) = A(t;t_0) + Q_1(t;t_0). \tag{3.4.8}$$

Using the subadditivity of σ_k we have

$$\sigma_k(Z(t)) \leq \sigma_k(A(t;t_0)) + \sigma_k(Q_1(t;t_0)). \tag{3.4.9}$$

Now, using $\rho_k(A)+\sigma_k(B) \leq \sigma_k(A+B) \leq \sigma_k(A)+\sigma_k(B)$ for symmetric matrices, we obtain

$$\sigma_k(Z(t)) \geq \rho_k(A(t;t_0)) + \sigma_k(Q_1(t;t_0)). \tag{3.4.10}$$

From (3.4.5) we get

$$\sigma_k(A(t;t_0) \leq \sigma_k(Z(t_0)) + \sigma_k\Big(\int_{t_0}^{t} Z^2(s)ds\Big).$$

Now using $|\sigma_k(A)|^2 \leq k\sigma_k(A^2)$ we get

$$|\sigma_k(Z(t_0))|^2 \leq k\sigma_k(Z^2(t_0)) < \epsilon^2.$$

We also have

$$\sigma_k\Big(\int_{t_0}^{t} Z^2(s)ds\Big) \leq Tr \int_{t_0}^{t} Z^2(s)ds < \epsilon.$$

Hence from (3.4.9), it follows that

$$\sigma_k(Z(t)) < 2\epsilon + \sigma_k(Q_1(t;t_0)) \tag{3.4.11}$$

for $t \geq t_0$. Now using the superadditivity of ρ_k, we have that

$$\rho_k(A(t_0;t)) \geq \rho_k(Z(t_0)) + \rho_k\Big(\int_{t_0}^{t} Z^2(s)ds\Big) > -\epsilon,$$

so that from (3.4.10), we obtain

$$\sigma_k(Z(t)) > -\epsilon + \sigma_k(Q_1(t;t_0)) \tag{3.4.12}$$

for $t \geq t_0$. We let $T_n \to \infty$ and $\tau_n \to \infty$ so that

$$\lim_{T_n \to \infty} \frac{1}{T_n} \int_{t_0}^{T_n} \sigma_k(Q_1(t;t_0))dt \equiv \eta_k(t_0) \tag{3.4.13}$$

and

$$\lim_{\tau_n \to \infty} \frac{1}{\tau_n} \int_{t_0}^{\tau_n} \sigma_k(Q_1(t;t_0)dt \equiv \mu_k(t_0). \tag{3.4.14}$$

From (3.4.11) we see that

$$\frac{1}{\tau_n} \int_{t_0}^{\tau_n} \sigma_k(Z(t))dt < \frac{2\epsilon(\tau_n - t_0)}{\tau_n} + \frac{1}{\tau_n} \int_{t_0}^{\tau_n} \sigma_k(Q_1(t;t_0))dt \tag{3.4.15}$$

and from (3.4.12) we obtain

$$\frac{1}{T_n} \int_{t_0}^{T_n} \sigma_k(Z(t))dt > \frac{-\epsilon(T_n - t_0)}{T_n} + \frac{1}{T_n} \int_{t_0}^{T_n} \sigma_k(Q_1(t;t_0)dt. \tag{3.4.16}$$

Now by the Schwartz inequality

$$\begin{aligned} \left|\frac{1}{T} \int_{t_0}^{T} \sigma_k(Z(t))dt\right|^2 &\leq \frac{1}{T} \int_{t_0}^{T} (\sigma_k(Z(t))^2 dt \\ &\leq \frac{k}{T} \int_{t_0}^{T} \sigma_k(Z^2(t))dt \\ &\leq \frac{k}{T} \int_{t_0}^{T} Tr\ Z^2(t)dt \to 0 \end{aligned}$$

as $T \to \infty$. Hence taking limits in (3.4.15) and (3.4.16) as $\tau_n \to \infty$, $T_n \to \infty$ we get

$$-\epsilon + \eta_k(t_0) \leq 0 \leq 2\epsilon + \mu(t_0)$$

which in turn gives

$$\eta_k(t_0) - \mu_k(t_0) < \delta,$$

a contradiction to (3.4.3), thus proving the theorem when both $\eta_k(t_0)$ and $\mu_k(t_0)$ are finite. Since $\sigma_k(A) \leq k\lambda_1(A)$ for all symmetric matrices A, in the case

$$\eta_k(t_0) = \limsup_{T\to\infty} \frac{1}{T}\int_{t_0}^{T} \sigma_k\Big(\int_{t_0}^{t} Q(s)ds\Big)dt = +\infty,$$

it follows that

$$\limsup_{T\to\infty} \int_{t_0}^{T} \lambda_1\Big(\int_{t_0}^{t} Q(s)ds\Big)dt = +\infty.$$

In this case the theorem follows by Theorem [3.2.2]. Finally, we assume that $\eta_k(t_0) < +\infty$ and $\mu_k(t_0) = -\infty$. That is,

$$\lim_{\tau_n} \frac{1}{\tau_n}\int_{t_0}^{\tau_n} \sigma_k(Q_1(t;t_0))dt = -\infty.$$

Hence, we get

$$-\epsilon + \eta_k(t_0) \leq 0 \leq 2\epsilon + \mu_k(t_0) = -\infty$$

a contradiction. This completes the proof in its entirety. □

THEOREM 3.4.3. *Assume $L(Q) > -\infty$. Then the matrix differential equation (3.4.2) is oscillatory if there exists $\delta > 0$ such that for all large t_0 we have*

$$\ell(t_0) + \delta \leq m(t_0)$$

where

$$\ell(t_0) = \lim_{t\to\infty} \text{approx inf } \lambda_1(Q_1(t;t_0));$$

$$m(t_0) = \lim_{t\to\infty} \text{approx sup } \lambda_1\Big(\int_{t_0}^{t} Q(s)ds\Big).$$

□

We can now state and prove a more general result :

THEOREM 3.4.4. *The matrix differential equation (3.4.2) is oscillatory if*

(i) $L(Q) > -\infty$;

(ii) *there exists* $\delta > 0$ *such that for all large* t_0, *we have*

$$m_k(t_0) - \ell_k(t_0) \geq \delta \tag{3.4.17}$$

for some $1 \leq k \leq n$, *where*

$$\ell_k(t_0) \equiv \lim_{t\to\infty} \text{approx inf } \sigma_k\Big(\int_{t_0}^{t} Q(s)ds\Big);$$

$$m_k(t_0) \equiv \lim_{t\to\infty} \text{approx sup } \sigma_k\Big(\int_{t_0}^{t} Q(s)ds\Big);$$

$$\sigma_k(A) = \sum_{i=1}^{k} \lambda_i(A).$$

PROOF: Assume that equation (3.4.2) is nonoscillatory. As before let $Y = Y(t)$ be a nontrivial prepared solution of (3.4.2) with $\det Y(t) \neq 0$ for $t \geq t_0$. Then $Z = -Y'Y^{-1}$ is a symmetric solution of the Riccati equation

$$Z(t) = Z(t_0) + \int_{t_0}^{t} Q(s)ds + \int_{t_0}^{t} Z^2(s)ds$$

for $t \geq t_0$. By defining $Q_1(t;t_0)$ and $A(t;t_0)$ as before and using Lemma [3.2.1] it is easy to see that $\lim_{t\to\infty} Tr\ A(t;t_0)$ exists. Taking $\epsilon > 0$ such that $\epsilon < \frac{\delta}{3}$,

as in the proof of the previous theorem, we may suppose that t_0 is sufficiently large so that

$$\int_{t_0}^{\infty} \sigma_k(Z^2(s))ds < \epsilon \tag{3.4.18}$$

$$|\sigma_k(Z(t_0))| < \epsilon$$

and

$$|\rho_k(Z(t_0))| < \epsilon.$$

Now using the Riccati equation and the subadditivity of σ_k we get

$$\begin{aligned}
\sigma_k(Z(t)) &\leq \sigma_k(A(t;t_0)) + \sigma_k(Q_1(t;t_0)) \\
&\leq \sigma_k(Z(t_0)) + \sigma_k\Big(\int_{t_0}^{t} Z^2(s)ds\Big) + \sigma_k(Q_1(t;t_0)) \\
&< \epsilon + \int_{t_0}^{t} \sigma_k(Z^2(s))ds + \sigma_k(Q_1(t;t_0)) \\
&< 2\epsilon + \sigma_k(Q_1(t;t_0)).
\end{aligned} \tag{3.4.19}$$

Similarly, using that fact that $\rho_k(A)+\sigma_k(B) \leq \sigma_k(A+B)$ for symmetric matrices A and B, we get

$$\begin{aligned}
\sigma_k(Z(t)) &\geq \rho_k(Z(t_0)) + \sigma_k\Big(\int_{t_0}^{t} Z^2(s)ds + Q_1(t;t_0)\Big) \\
&\geq \rho_k(Z(t_0)) + \rho_k\Big(\int_{t_0}^{t} Z^2(s)ds\Big) + \sigma_k(Q_1(t;t_0)) \\
&> -\epsilon + \sigma_k(Q_1(t;t_0)).
\end{aligned} \tag{3.4.20}$$

For a symmetric matrix A, we know that

$$|\rho_k(A)|^2 = |\sigma_k(-A)|^2 \leq k\sigma_k(A^2).$$

Therefore it follows that $\sigma_k(Z(t))$ cannot be $\leq -\eta$ on a set of infinite measure and $\sigma_k(Z(t))$ cannot be $\geq \eta$ on a set of infinite measure for any $\eta > 0$. Hence we get

$$\begin{aligned} \lim_{t\to\infty} \text{approx } \sigma_k(Z(t)) &= 0 \\ &= \lim_{t\to\infty} \text{approx inf } \sigma_k(Z(t)) \\ &= \lim_{t\to\infty} \text{approx sup } \sigma_k(Z(t)). \end{aligned} \tag{3.4.21}$$

Therefore, from (3.4.19) we have

$$\begin{aligned} 0 &\leq \lim_{t\to\infty} \text{approx inf}(\sigma_k(A(t;t_0)) + \sigma_k(Q_1(t;t_0)) \\ &\leq 2\epsilon + \lim_{t\to\infty} \text{approx inf } \sigma_k(Q_1(t;t_0)) \\ &= 2\epsilon + \ell_k(t_0). \end{aligned} \tag{3.4.22}$$

Similarly, it follows from (3.4.20) that

$$\begin{aligned} 0 &\geq \lim_{t\to\infty} \text{approx sup } \Big(\rho_k(Z(t_0)) + \sigma_k\Big(\int_{t_0}^{t} Z^2(s)ds + Q_1(t;t_0)\Big)\Big) \\ &\geq -\epsilon + \lim_{t\to\infty} \text{approx sup } \sigma_k(Q_1(t;t_0)) \\ &= -\epsilon + m_k(t_0). \end{aligned} \tag{3.4.23}$$

Combining (3.4.22) and (3.4.23) we get

$$m_k(t_0) - \ell_k(t_0) < \delta,$$

a contradiction. Hence (3.4.2) is oscillatory.

CHAPTER IV

LEIGHTON-TYPE RESULTS FOR SYSTEMS

4.1. Introduction

For the second-order scalar differential equation

$$(p(t)x')' + q(t)x = 0 \tag{4.1.1}$$

where $p(t) > 0$, $q(t) \in C[a, +\infty)$, many oscillation criteria have been developed [26, 52, 56, 58]. A well known result due to Leighton [39], generalizes the earlier oscillation criterion of Fite-Wintner which states that the second order scalar differential equation

$$x'' + q(t)x = 0 \tag{4.1.2}$$

is oscillatory in case the coefficient $q(t)$ satisfies the condition

$$\int_a^\infty q(s)ds = +\infty. \tag{4.1.3}$$

LEIGHTON'S THEOREM. *The second-order scalar differential equation (4.1.1) is oscillatory if*

$$\int_a^\infty p^{-1}(s)ds = \int_a^\infty q(s)ds = +\infty. \tag{4.1.4}$$

□

It is of interest to look for analogues of these results for systems.

4.2. An Example

Consider the corresponding self-adjoint matrix differential equation

$$(P(t)Y')' + Q(t)Y = 0 \tag{4.2.1}$$

where P, Q, Y are $n \times n$ real continuous matrix valued functions with $P(t), Q(t)$ symmetric and $P(t)$ positive definite for $t \in [a, +\infty)$. We also consider the formally self-adjoint matrix differential equation

$$Y'' + Q(t)Y = 0 \tag{4.2.2}$$

where $Q(t)$ is as above.

We shall be interested in an analogue of Leighton's theorem for equation (4.2.1). Consider the following simple example [7]): Let $P(t) = Q(t) = \text{diag}(e^{2t}, e^{-2t})$ in (4.2.1). That is, consider the 2-dimensional system

$$\left(\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' \right)' + \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0.$$

Then a nontrivial prepared nonoscillatory solution is $Y(t) = \text{diag}(e^{-t}, e^{t})$. However, one can see that

(i) $\lambda_1\left(\int_0^t P^{-1}(s)ds\right)$ and $\lambda_1\left(\int_0^t Q(s)ds\right)$ both tend to $+\infty$ as $t \to \infty$;

(ii) $\int_0^t \det(P^{-1}(s))ds$ and $\int_0^t \det(Q(s))ds$ both tend to $+\infty$ as $t \to +\infty$;

(iii) $\int_0^t Tr(P^{-1}(s))ds$ and $\int_0^t Tr\ (Q(s))ds$ both tend to $+\infty$ as $t \to +\infty$.

In the light of the above example, it is not obvious what the analogous theorem for system (4.2.1) should be.

4.3. A Conjecture

Using positive linear functionals Etgen and Pawlowski [20] have shown that equation (4.2.2) is oscillatory at ∞ if there exists a positive linear functional ϕ on the set of all $n \times n$ real symmetric matrices such that

$$\lim_{t\to\infty} \phi\Big(\int_0^t Q(s)ds\Big) = \infty. \tag{4.3.1}$$

As the trace of a matrix is a positive linear functional, it immediately follows that equation (4.2.2) is oscillatory at ∞ whenever

$$\lim_{t\to\infty} Tr\Big(\int_0^t Q(s)ds\Big) = +\infty. \tag{4.3.2}$$

There are several results dealing with the oscillation of the equation (4.2.1), and they are largely based on the use of positive linear functionals [1, 7, 19, 20, 21, 28, 54]. The basic result obtained is that equation (4.2.1) is oscillatory on $[0,\infty)$ in case there exists a positive linear functional ϕ such that the scalar equation

$$(\phi(P(t))y')' + \phi(Q(t))y = 0$$

is oscillatory, where it is assumed, without loss of generality, that ϕ is normalized so that $\phi(I) = 1$.

Although condition (4.3.2) implies the oscillation of equation (4.2.2), Mingarelli [45], observed that the condition

$$\liminf_{t\to\infty} Tr\Big(\int_0^t Q(s)ds\Big) = -\infty \tag{4.3.3}$$

may or may not imply oscillation of the equation (4.2.2).

It has been conjectured (see Hinton and Lewis [28]) that the equation (4.2.2) is oscillatory at ∞ if

$$\lim_{t\to\infty} \lambda\{Q_1(t)\} = +\infty$$

where $Q_1(t) = \int_0^t Q(s)ds$.

In the case of a non-negative definite coefficient matrix $Q(t)$, using the fact that for such $Q(t)$,

$$\lambda_1\Big(\int_s^t Q(s)ds\Big) \le Tr\Big(\int_0^t Q(s)ds\Big)$$

Mingarelli proved that equation (4.2.2) is oscillatory. He further showed that the conjecture is also true in the case when $Q(t)$ is a constant matrix, namely,

THEOREM 4.3.1. *Let Q be a symmetric constant matrix. Then*

(A) Equation (4.2.2) is oscillatory at ∞ if Q possesses at least one positive eigenvalue.

(B) If $\lambda_1(Q) \le 0$, the equation (4.2.2) is non-oscillatory, and in fact, disconjugate on $[0,\infty)$.

□

Mingarelli [45] also proved that the conjecture is true under the additional assumption that

$$\liminf_{t\to\infty} \frac{1}{t} Tr\Big(\int_0^t Q(s)ds\Big) > -\infty.$$

He [44] later replaced the above condition by

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t Tr\Big(\int_0^s Q(\sigma)d\sigma\Big)ds > -\infty.$$

These results were further generalized by Akiyama, Kaper, Kwong and Mingarelli [38]. Butler and Erbe [6] showed that the conjecture is true even if (4.3.4) does not hold, provided a weaker condition than (4.3.4) holds. In fact, they proved:

THEOREM 4.3.2 [6]. *Let $g(t)$ be a positive, absolutely continuous, real valued function which is nondecreasing on $[a,\infty)$ and assume*

(i) $\limsup\text{approx}\inf_{t\to\infty} \frac{1}{g(t)} Tr\Big(\int_a^t Q(s)ds\Big) = \ell > -\infty;$

(ii) $\limsup\text{approx}\inf_{t\to\infty} \frac{1}{g(t)} \int_a^t \Big(\lambda_1 \int_a^s Q(\sigma)d\sigma)\Big)^2 ds = +\infty$

and that $\lim_{t\to\infty} \lambda_1\Big(\int_a^t Q(s)ds\Big) = +\infty.$

Then (4.2.2) is oscillatory.

□

It was also shown by Butler and Erbe that the condition

$$\lim_{t\to\infty} \lambda_1\Big(\int_a^t Q(s)ds\Big) = +\infty$$

can also be weakened somewhat provided a certain relation holds between the largest and smallest eigenvalues of $\int_a^t Q(s)ds$ as $t \to \infty$.

THEOREM 4.3.3 [7]. *Equation (4.2.2) is oscillatory if one of the following set of hypotheses holds:*

(1) (a) $\limsup\text{approx}\inf_{t\to\infty} \lambda_1\Big(\int_a^t Q(s)ds\Big) = \infty$ and

(b) $\limsup\text{approx}\sup_{t\to\infty} \left|\frac{\lambda_1(Q_1(t,a))}{\lambda_n(Q_1(t,a))}\right| > 0;$

or

(2) (a) $\limsup\limits_{t\to\infty}$ approx $\lambda_1(Q_1(t,a)) = \infty$ and

(b) $\liminf\limits_{t\to\infty}$ approx $\left|\dfrac{\lambda_1(Q_1(t,a))}{\lambda_n(Q_1(t,a))}\right| > 0.$

□

Using the behaviour of the other eigenvalues, Atkinson, Kaper and Kwong [2] proved

THEOREM 4.3.4 [2]. *Equation (4.2.2) is oscillatory at infinity if*

$$\lim_{t\to\infty} \lambda_i\Big\{\int_a^t Q(s)ds\Big\} = \infty, \quad i = 1,2,\ldots,n-1.$$

□

Kaper and Kwong in [36] first gave an affirmative answer to the conjecture when the dimension n equals 2. Finally, Kaper and Kwong [37] showed that the equation (4.2.2) is oscillatory if the largest eigenvalue of the matrix $\int_a^t Q(s)ds$ is sufficiently large on a sufficiently large set of t-values, thus confirming and strengthening the conjecture.

THEOREM 4.3.5 [37]. *There exists a finite number T_n, whose value depends on the dimension n, such that equation (4.2.2) is oscillatory at infinity whenever*

$$\liminf_{\alpha\to\infty}\{\alpha \ mes \ S_\alpha(a,\infty)\} \geq T,$$

where $S_\alpha(a,\infty) = \Big\{t \in (a,\infty) : \lambda_1\Big(\int_a^t Q(s)ds\Big) \geq \alpha\Big\}$ *for any positive number* α.

Using the method of Kaper and Kwong [37], we can obtain a similar result for the equation (4.2.1), in case $P(t)$ *is diagonal such that* $P(t) \leq I$.

Taking $R(t) = -P(t)Y'(t)Y^{-1}(t)$ and using the fact that $Y(t)$ is a solution of (4.2.1) we get the Riccati equation

$$R(t) = R(a) + \int_a^t Q(s)ds + \int_a^t R(s)P^{-1}(s)R(s)ds$$

which can be written as

$$R(t) = F(t) + \int_a^t R(s)P^{-1}(s)R(s)ds \tag{4.3.4}$$

where

$$F(t) = R(a) + Q_1(t) \tag{4.3.5}$$

and $Q_1(t) = \int_a^t Q(s)ds$. For $\alpha > 0$, define S_α as

$$S_\alpha(a,b) = \{t \in (a,b) : \lambda_1\{F(t)\} \geq \alpha\}.$$

LEMMA 4.3.6. *There exists T_n such that*

$$\int_a^b R(s)P^{-1}(s)R(s)ds \geq \frac{1}{10}I \tag{4.3.6}$$

whenever mes $S_1(a,b) \geq T_n$.

PROOF: Lemma 4.3.6 will be proved by using induction on the dimension n. When $n = 1$, we are dealing with the scalar case. In this case $\lambda_1\{F(t)\} = F(t)$ and we also have

$$\int_a^b R(s)P^{-1}(s)R(s)ds = \int_a^b P^{-1}R^2(s)ds.$$

As $P^{-1}(t) > 0$, $R^2(t) \geq 0$, it follows from (4.3.4), using $F(t) \geq 1$, that $R(t) \geq 1$. Hence

$$\begin{aligned}\int_a^b R(s)P^{-1}(s)R(s)ds &= \int_a^b P^{-1}(s)R^2(s)ds \\ &\geq P^{-1}(t_*)\int_a^b R^2(s)ds \\ &\geq P^{-1}(t_*)mes\ S_1(a,b)\end{aligned}$$

where $t_* \in [a,b]$ such that $P^{-1}(t_*) = \min_{t\in[a,b]} P^{-1}(t)$. We can take $T_1 = \frac{1}{10}$ and the lemma is true.

Now assume that Lemma 4.3.6 is true for dimensions less than or equal to n. For $k \leq n$ we can establish a number of results, some of which show that the constant $\frac{1}{10}$ can be dropped if the set S_α is large enough and others establish the relation between the size $S_\alpha(a,b)$ and nonexistence of continuous solutions to Riccati equations. In fact, using change of variables and translation we obtain

LEMMA 4.3.7. *If* $mes\ S_\alpha(a,b) \geq \frac{10}{\alpha}T_n$ *then*

$$\int_a^b R(s)P^{-1}(s)R(s)ds \geq \alpha I.$$

The connection between $S_\alpha(a,b)$ and the nonexistence of a continuous solution to the Riccati equation is established in the following

LEMMA 4.3.8. *The Riccati equation (4.3.4) does not have a continuous solution on* $[a,b)$ *if mes* $S_1(a,b) \geq 20T_n$.

If possible, assume the contrary. Taking $a_1 = a$ and $b_1 < b$ such that *mes* $S_1(a_1,b_1) = 10\ T_n$ we get

$$\int_{a_1}^{b_1} R(s)P^{-1}(s)R(s)ds \geq I.$$

Now let $a_2 = b_1$ and choose b_2 such that *mes* $S_1(a_3,b_3) = \frac{10\ T_n}{2}$. Consider

$$R(t) = F_2(t) + \int_{a_2}^{t} R(s)P^{-1}(s)R(s)ds, \quad t \geq a_2$$

where $F_2(t) = F(t) + \int_{a_1}^{b_1} R(s)P^{-1}(s)R(s)ds$. Note that [5, 42]

$$\lambda_1\{F_2(t)\} \geq \lambda_1\{F(t)\} + \lambda_n\Big\{\int_{a_1}^{b_1} R(s)P^{-1}(s)R(s)ds\Big\}$$

$$\geq 2 \quad \text{for all } t \in S_1(a_2,b_2).$$

Hence by the previous lemma we get

$$\int_{a_2}^{b_2} R(s)P^{-1}(s)R(s)ds \geq 2I.$$

Now let $a_3 = b_2$ and choose b_3 such that *mes* $S_1(a_3,b_3) = \frac{10T_n}{4}$. Consider

$$R(t) = F_3(t) + \int_{a_3}^{t} R(s)P^{-1}(s)R(s)ds, \quad t \geq a_3,$$

where

$$F_3(t) = F(t) + \int_{a_1}^{b_1} R(s)P^{-1}(s)R(s)ds + \int_{a_2}^{b_2} RP^{-1}(s)R(s)ds.$$

Note that $\lambda_1\{F_3(t)\} \geq 4$ for all $t \in S_1(a_3, b_3)$. Hence we obtain

$$\int_{a_3}^{b_3} R(s)P^{-1}(s)ds \geq 4I.$$

Note that $b_n \to \bar{b} \leq b$ where $\bar{b}$ is such that $mes\ S_1(a, \bar{b}) = \frac{\pi^2}{6} 10\ T_n$ and $\int_a^t R(s)P^{-1}(s)R(s)ds$ exceeds any multiple of I as $t \to \bar{b}$. Hence lemma (4.3.8) follows.

LEMMA 4.3.9. *If mes* $S_\alpha(a, b) \geq \frac{20\ T_n}{\alpha}$, *then the Riccati equation (4.3.4) does not have a continuous solution on* $[a, b)$.

Assume the contrary. That is, we assume that

$$R(t) = F(t) + \int_a^t R(s)P^{-1}(s)R(s)ds$$

exists as a continuous solution on $[a, b)$. Using $\bar{R}(t) = \frac{1}{\alpha} R\left(\frac{t}{\alpha}\right)$ we get

$$\bar{R}(t) = \bar{F}(t) + \int_{\alpha a}^t \bar{R}(s)\bar{P}^{-1}(s)\bar{R}(s)ds$$

exists as a continuous solution on $[\alpha a, \alpha b)$, where $\bar{F}(t) = \frac{1}{\alpha} F\left(\frac{t}{\alpha}\right)$ and $\bar{P}(t) = P\left(\frac{s}{\alpha}\right)$. Letting $\bar{S}(\alpha a, \alpha b) = \{t \in (\alpha a, \alpha b) : \lambda_1\{\bar{F}(t)\} \geq 1\}$ note that $mes\ \bar{S}(\alpha a, \alpha b) = \alpha\ mes\ S_\alpha(a, b) \geq 20\ T_n$. Hence $\bar{R}(t)$ is not a continuous solution, a contradiction.

LEMMA 4.3.10. *If mes* $S_1(a, b) \geq \frac{20\ T_n}{\delta^2}$, $\delta > 0$, *then*

$$R(t) = \delta\left[F(t) + \int_a^t R(s)P^{-1}(s)R(s)ds\right]$$

does not have a continuous solution on $[a, b)$.

To prove this, let $\bar{R}(t) = \frac{1}{\delta} R\left(\frac{t}{\delta^2}\right)$ and note that $\bar{R}(t)$ satisfies

$$\bar{R}(t) = \bar{F}(t) + \int_{\delta^2 a}^{t} \bar{R}(s)\bar{P}^{-1}(s)\bar{R}(s)ds, \quad [\delta^2 a, \delta^2 b)$$

where $\bar{P}(s) = P\left(\frac{s}{\delta^2}\right)$ and $\bar{F}(s) = F\left(\frac{s}{\delta^2}\right)$, if $R(t)$ satisfy (4.3.4). Defining

$$\bar{S}_1(\delta^2 a, \delta^2 b) = \{t \in (\delta^2 a, \delta^2 b) : \lambda_1\{\bar{F}(t)\} \geq 1\}$$

note that $mes\ \bar{S}_1(\delta^2 a, \delta^2 b) = \delta^2 mes\ S_1(a, b)$. As long as $R(t)$ is a continuous solution on $[a, b)$, $\bar{R}(t)$ is a continuous solution on $[\delta^2 a, \delta^2 b)$. Hence we get the conclusion.

Now we continue the proof of Lemma 4.3.6. Assume that Lemma 4.3.6 is not true for $n + 1$, where we take $T_{n+1} = 2000T_n + 40n^2(n + 1)$. Hence there exists a right neighborhood $[a, b)$ of a where $mes\ S_1(a, b) \geq T_{n+1}$, but the solution of (4.3.4) does not satisfy the ordering relation (4.3.6). Without loss of generality we may assume that

$$\int_a^b R(s)P^{-1}(s)R(s)ds = \ \mathrm{diag}(\delta_1, \delta_2, \ldots, \delta_{n+1})$$

where $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_{n+1}$. Then $\delta_{n+1} < \frac{1}{10}$. Using the following notation,

$$R(t) = [r_{ij}]_{(n+1)\times(n+1)}, \ G(t) = \int_a^t R(s)P^{-1}(s)R(s)ds = [\rho_{ij}(t)]_{(n\times 1)(z+1)},$$

and using the fact that $P(t)$ is diagonal and $R^T = R$, we get

$$\sigma_{n+1} = \sum_{j=1}^{n+1} \int_1^b r_{n+1,j}^2(t)p_{jj}(t)dt < \frac{1}{10}.$$

Hence we get

$$\int_a^b r_{n+1,j}^2(t)p_{jj}^{-1}(t)ds < \frac{1}{10} \quad \text{for} \quad j = 1,2,\ldots,n+1. \tag{4.3.7}$$

Using these inequalities which put an upper bound on the measure of the set of points t in (a,b) where $\sqrt{r_{n+1,j}^2(t)p_{jj}^{-1}(t)}$ exceeds any given value we obtain

$$mes\Big\{t \in (a,b) : \sqrt{r_{n+1,j}^2(t)p_{jj}^{-1}(t)} \geq \frac{1}{20n}\Big\} < 40n^2. \tag{4.3.8}$$

If (4.3.8) is not true then we get

$$\int_a^b r_{n+1,j}^2(t)p_{jj}^{-1}(t)dt \geq \frac{1}{400n^2} \cdot 40n^2 \geq \frac{1}{10}$$

which contradicts (4.3.7). Thus, if we eliminate from $S_1(a,b)$ all those points t where

$$\sqrt{r_{n+1,j}^2(t)p_{jj}^{-1}(t)} \geq \frac{1}{20n}$$

for some j and call the remaining set $S_1^{\#}(a,b)$,

$$S_1^{\#}(a,b) = \Big\{t \in (a,b) : \lambda_1\{F(t)\} \geq 1 \quad \text{and} \quad \sqrt{r_{n+1,j}^2(t)P_{jj}^{-1}(t)} < \frac{1}{20n}, \quad j = 1,2,\ldots,n+1\Big\},$$

then, as $mes\ S_1(a,b) \geq T_{n+1}$, using (4.3.8) we get

$$mes\ S_1^{\#}(a,b) > T_{n+1} - 40n^2(n+1) = 2000T_n.$$

We introduce the following partitions of $R(t)$, $G(t)$ and $P(t)$:

$$R(t) = \begin{bmatrix} R_{n\times n}^{\#}(t) & r_{n\times 1}^{\#}(t) \\ \left(r^{\#^T}(t)\right)_{1\times n} & \alpha(t) \end{bmatrix}, \qquad P(t) = \begin{bmatrix} P_{n\times n}^{\#}(t) & 0_{n\times 1} \\ 0_{1\times n} & p_{n+1,n+1}^{-1}(t) \end{bmatrix}$$

$$G(t) = \begin{bmatrix} G_{n\times n}^{\#}(t) & g_{n\times 1}^{\#}(t) \\ \left(g^{\#^T}(t)\right)_{1\times n} & \beta(t) \end{bmatrix}$$

and noting that

$$G(t) = \int_a^t \begin{bmatrix} R^{\#}_{n\times n} & r^{\#}_{n+1} \\ r^{\#^T}_{1\times n} & \alpha(s) \end{bmatrix} \begin{bmatrix} P^{\#^{-1}}_{n\times n} & 0 \\ 0 & p_{n+1,n+1} \end{bmatrix} \begin{bmatrix} R^{\#}_{n\times n}(s) & r^{\#}_{n\times 1}(s) \\ r^{\#^T}_{1\times n}(s) & \alpha(s) \end{bmatrix} ds,$$

we have, in $n \times n$ dimensions

$$G^{\#}(t) = \int_a^t R^{\#} P^{\#^{-1}} R^{\#}(s) ds + \int_a^t r^{\#} p^{-1}_{n+1,n+1} r^{\#^T} ds.$$

By taking

$$A(t) = \begin{bmatrix} R^{\#} - \frac{1}{10}\int_a^t R^{\#} P^{\#^{-1}} R^{\#} ds + \frac{9}{10} I_n & 0_{n\times 1} \\ 0_{1\times n} & \frac{19}{20} \end{bmatrix}$$

$$U(t) = \begin{bmatrix} \frac{1}{10}\int_a^t r^{\#}_{n\times 1} p^{-1}_{n+1,n+1} r^{\#^T}_{1\times n} ds & 0_{n\times 1} \\ 0_{1\times n} & 0_{1\times 1} \end{bmatrix}$$

$$V(t) = \begin{bmatrix} \frac{9}{10} G^{\#} & g^{\#} \\ g^{\#^T} & \frac{8}{10} + \beta(t) \end{bmatrix}, \quad W(t) = \begin{bmatrix} \frac{9}{10} I_n & -r^{\#}(t) \\ -r^{\#^T} & \frac{3}{20} - \alpha(t) \end{bmatrix}$$

we see that

$$F(t) = R(t) - G(t) = A(t) - U(t) - V(t) - W(t).$$

From the definition $U(t) \geq 0$ for all $t \in [a,b]$. Now from the definition of $S_1^{\#}(a,b)$ we have, on $S_1^{\#}(a,b)$ that

$$\sqrt{r^2_{n+1,j}(t) p^{-1}_{jj}(t)} < \frac{1}{20n}, \quad j = 1,2,\ldots,n$$

$$|\alpha(t)| = \left| r_{n+1,n+1}(t) \right| < \sqrt{r^2_{n+1,n+1}(t) p^{-1}_{n+1,n+1}(t)} < \frac{1}{20n} < \frac{1}{20}$$

$$\beta(t) = \int_a^t \sum_{i=1}^{n+1} r^2_{n+1,i} P^{-1}_{ii} = \rho_{n+1,n+1}(t) \leq \rho(b) = \sigma_{n+1} < \frac{1}{10}.$$

For any $(n+1)$-dimensional vector $x = (x_1, x_2, \ldots, x_{n+1})^T = (x^{\#}, x_{n+1})^T$ we have

$$\begin{aligned}(x, V(t)x) &= \frac{9}{10}\langle x^{\#}, G^{\#}(t)x^{\#}\rangle + 2\langle x^{\#}, g^{\#}(t)x_{n+1}\rangle \\ &\quad + \Big(\frac{8}{10} + \beta(t)\Big)x_{n+1}^2 \\ &> \frac{1}{9}\langle x^{\#}, G^{\#}(t)x^{\#}\rangle + 2\langle x^{\#}, g^{\#}(t)x_{n+1}\rangle \\ &\quad + 9\beta(t)x_{n+1}^2 + 8\Big(\frac{1}{10} - \beta(t)\Big)x_{n+1}^2 \\ &= \langle y, G(t)y\rangle + 8\Big(\frac{1}{10} - \beta(t)\Big)x_{n+1}^2\end{aligned}$$

where $y = \Big(\frac{1}{3}x^{\#}, 3x_{n+1}\Big)^T$. Also we have

$$\begin{aligned}(x, w(t)x) &= \frac{9}{10}\|x^{\#}\|^2 - 2\langle x^{\#}, r^{\#}x_{n+1}\rangle + \Big(\frac{3}{20} - \alpha(t)\Big)x_{n+1}^2 \\ &\geq \frac{1}{10}\|x\|^2 - 2\sum_{j=1}^{n}|r_{j,n+1}|\ \|x\|^2 + \Big(\frac{1}{20} - \alpha(t)\Big)x_{n+1}^2\end{aligned}$$

using the inequalities which hold on $S_1^{\#}(a,b)$ we get $V(t) \geq 0$ and $w(t) \geq 0$ for all $t \in S_1^{\#}(a,b)$.

Taking $Z(t) = U(t)+V(t)+W(t)$ we have $Z(t) \geq 0$ on $S_1^{\#}(a,b)$ and $A(t) = F(t)+Z(t)$. Since $\lambda_1\{F(t)\} \geq 0$ on the set $S_1(a,b)$, it follows that $\lambda_1\{A(t)\} \geq 1$ on $S_1^{\#}(a,b)$. From the definition of $A(t)$, we see that any eigenvalue of $A(t)$ that is greater than 1 must also be an eigenvalue of the $n \times n$ submatrix in the top left corner, in the definition of $A(t)$. Hence for $t \in S_1^{\#}(a,b)$,

$$\lambda_1\Big\{R^{\#} - \frac{1}{10}\int_a^t R^{\#}P^{\#^{-1}}R^{\#}(s)ds + \frac{9}{10}I_n\Big\} \geq 1. \qquad (4.3.9)$$

Let $F^{\#}$ be defined in terms of $R^{\#}$ by

$$F^{\#}(t) = 10\Big[R^{\#} - \frac{1}{10}\int_a^t R^{\#}P^{\#^{-1}}R^{\#}(s)ds\Big]. \tag{4.3.10}$$

Then it follows from (4.3.9) that

$$\lambda_1\{F^{\#}(t)\} \geq 1, \quad t \in S^{\#}(a,b).$$

This leads to the contradiction. We can interpret the definition of $F^{\#}$ as a matrix Riccati equation for $R^{\#}$,

$$R^{\#}(t) = \frac{1}{10}\Big[F^{\#}(t) + \int_a^t R^{\#}P^{\#^{-1}}R^{\#}(s)ds\Big] \tag{4.3.11}$$

and then the existence of continuous solution $R(t)$ of (4.3.4) on $[a,b)$ implies the existence of a continuous solution $R^{\#}$ of (4.3.11) on $[a,b)$. On the other hand, because $mes\ S^{\#}(a,b) > 2000T_n$, (4.3.11) can not have a continuous solution on $[a,b)$. Hence the proof follows by induction. □

Now define

$$S_\alpha(a,\infty) = \{t \in (a,\infty) : \lambda\{Q_1(t)\} \geq \alpha\}$$

for any positive α. We then have

THEOREM 4.3.11. *There exists a finite number $T(n)$ such that*

$$(P(t)Y'(t))' + Q(t)Y(t) = 0 \tag{4.3.12}$$

is oscillatory at infinity whenever

$$\lim_{\alpha\to\infty} \inf\{\alpha\ mes\ S_\alpha(a,\infty)\} \geq T(n). \tag{4.3.13}$$

PROOF: Let $T(n) = 80T_n$, where T_n is the number guaranteed by Lemma 4.3.6. From (4.3.13) we get that $mes\ S_\alpha(a,\infty) \geq \left(\frac{40}{\alpha}\right)T_n$ for α sufficiently large. Furthermore we also have, when α is sufficiently large and $\lambda_1\{Q_1(t)\} \geq \alpha$ for some t, that $\lambda_1\{R(a) + Q_1(t)\} \geq \frac{\alpha}{2}$. Hence

$$mes\{t \in (a,\infty) : \lambda_1\{R(a) + Q_1(t)\} \geq \frac{\alpha}{2}\} \geq \frac{40}{\alpha} \cdot T_n.$$

But then by the results established as a consequence of Lemma (4.3.6), we get that the equation (4.3.12) does not have a nonsingular conjoined solution on $[a,\infty)$ and hence is oscillatory at infinity.

□

The above theorem expresses the fact that oscillation at infinity is possible whenever $\lambda_1\{Q(t)\}$ is sufficiently large on a sufficiently large set. The conjecture guarantees this phenomenon in the case of the formally self-adjoint equation

$$Y''(t) + Q(t)Y(t) = 0. \tag{4.3.14}$$

COROLLARY 4.3.12. *If there is a measurable subset J of $[a,\infty)$ of infinite measure, such that*

$$\lim_{t\to\infty} \lambda_1\{Q_1(t)\} = \infty$$

then (4.3.14) is oscillatory at infinity.

4.4. Different Proof of the Conjecture

In this section we are specifically concerned with the self adjoint matrix differential equation

$$(P(t)Y')' + Q(t)Y = 0 \tag{4.4.1}$$

where P, Q are real, symmetric, continuous matrix valued functions with $P(t)$ positive definite. In addition to the condition

$$\lim_{t\to\infty} \lambda_1\Big(\int_0^t Q(s)ds\Big) = \infty \tag{4.4.2}$$

which comes from the corresponding formally self-adjoint system

$$Y'' + Q(t)Y = 0, \tag{4.4.3}$$

various conditions on $P(t)$ (namely on the eigenvalues of $P(t)$) are required, as the following discussion will show, in order to conclude that the equation (4.4.1) is oscillatory. The following are some such results by Butler and Erbe:

THEOREM 4.4.1. *The equation (4.4.1) is oscillatory if*

(i) $\lim_{t\to\infty} \lambda_1\Big(\int_a^t Q(s)ds\Big) = \infty;$

(ii) $\lim_{t\to\infty} \int_a^t \lambda_n(P^{-1}(s))ds = \infty;$

and

(iii) $\liminf_{t\to\infty} \dfrac{\lambda_1\Big(\int_a^t Q(s)ds\Big)}{\lambda_n\Big(\int_a^t Q(s)ds\Big)} > 0.$

□

The following result extends a result due to Mingarelli [45]:

THEOREM 4.4.2 [7]. *Let $g(t)$ be a positive, absolutely continuous, nondecreasing function on $[a,\infty)$ and assume*

(i) $\liminf_{t\to\infty} \dfrac{1}{g(t)}\, Tr\Big(\int_a^t Q(s)ds\Big) > -\infty;$

(ii) $\lim_{t\to\infty} \dfrac{1}{g(t)} \int_a^t \lambda_n(P^{-1}(s))\Big[\lambda_1 \int_a^s Q(\sigma)d\sigma\Big]^2 ds = +\infty;$

and

(iii) $\lim_{t\to\infty} \lambda_1\Big(\int_a^t Q(s)ds\Big) = \lim_{t\to\infty} \int_a^t \lambda_n(P^{-1}(s))ds = +\infty;$

then equation (4.4.1) is oscillatory.

□

Using principal submatrices [5] and functionally commutative matrices [Chapter 5] Butler and Erbe also proved

THEOREM 4.4.3 [7]. *Let $g(t)$ be as in the above theorem and assume $P(t)$ is functionally commutative on $[a,\infty)$. Assume further that there exists k, $1 \le k < n$ such that*

(i) $\liminf_{t\to\infty}\Big(\frac{1}{g(t)}\Big)Tr\Big(\int_a^t Q_k(s)ds\Big) > -\infty;$

(ii) $\lim_{t\to\infty}\Big(\frac{1}{g(t)}\Big)\int_a^t \lambda_k[(P^{-1}(s))_k]\Big[\lambda\Big(\int_a^s Q_k(\sigma)\Big)\Big]^2 ds = +\infty;$

and

(iii) $\lim_{t\to\infty} \lambda_1\Big(\int_a^t Q_k(s)ds\Big) = \lim_{t\to\infty} \int_a^t \lambda_k[(P^{-1}(s))_k]ds = +\infty.$

Then equation (4.4.1) is oscillatory.

□

Recently Byers, Harris and Kwong [10] proved

THEOREM 4.4.4 [10]. *The equation (4.4.1) is oscillatory if*

$$\limsup_{\lambda\to\infty} \lambda\Big(\int_{S_\lambda} \lambda_n(P^{-1}(t))dt\Big) > n$$

where

$$S_\lambda = \Big\{t \in [0,\infty) : \lambda_1\Big(\int_0^t Q(s)ds\Big) \ge \lambda\Big\}.$$

□

The above Theorem also proves the conjecture.

Coles [12] gave some more oscillation criteria for (4.4.1) by using the notion of weighted means introduced by Byers, Harris and Kwong [10]. Let $f_1, f_2, \ldots, f_N$ be non-negative, locally integrable scalar-valued functions defined on $[0, \infty)$. For a matrix-valued function $H(t)$ and $t \geq a \geq a$, define

$$I_1(t, a; H) = \int_a^t f_1(s)H(s)ds$$

$$J_1(t, a; H) = \int_a^t f_1^2(s)H(s)ds$$

$$i_1(t, a) = \int_a^t f_1(s)ds.$$

For $2 \leq k \leq N$:

$$I_k(t, a; H) = \int_a^t f_k(s)I_{k-1}(s, a; H)ds$$

$$J_k(t, a; H) = \int_a^t f_k^2(s)f_{k-1}^{-1}(s)J_{k-1}(s, a; H)ds$$

$$i_k(t, a) = \int_a^t f_k(s)i_{k-1}(s)ds.$$

For $1 \leq k \leq N$:

$$I_k(t; H) = I_k(t, 0, H)$$

$$J_k(t; H) = J_k(t, 0; H)$$

$$i_k(t) = i_k(t, 0).$$

Let

$$S_\lambda = \left\{ t \geq 0; \lambda_1 \left\{ i_n^{-1}(t) I_N\left(t; \int_0^t Q(s)ds\right)\right\} \geq \lambda \right\}$$

$$S_\lambda(t) = S_\lambda \cap [t, \infty)$$

$$L(\lambda) = \limsup_{t\to\infty} \; i_N(t) \cdot \int_{S_\lambda(t)} f_N(s)\lambda_n\{J_N^{-1}(s;P\}ds.$$

Now we can state the results of Coles:

THEOREM 4.4.5. *Equation (4.4.1) is oscillatory if*

$$\limsup_{\lambda\to\infty} \; \lambda \int_{S_\lambda} f_N(s)\lambda_n\{J_N^{-1}(s;P\}ds > \frac{n}{i_N(\infty)}.$$

THEOREM 4.4.6. *Equation (4.4.1) is oscillatory if*

$$\limsup_{\lambda\to\infty} \; \lambda L(\lambda) > n.$$

4.5. Leighton-type results for systems

By relaxing the stringent conditions on the smallest eigenvalue of $P(t)$, by going to some other intermediate eigenvalue of $P(t)$, and by strengthening the conditions on the intermediate eigenvalues of $Q(t)$ one can obtain an oscillation result for equation (4.4.1). This result can be viewed as a proper generalization of Leighton's result mentioned earlier in this chapter. The method of proof is modeled along the techniques used in Byers, Harris and Kwong [10] and the result obtained includes Theorem 4.4.4 as a special case when $k = 1$.

THEOREM 4.5.1. *The equation (4.4.1) is oscillatory if there exists* k, $1 \leq k \leq n$ *such that*

$$\limsup_{\lambda \to \infty} \lambda \int_{S_\lambda^k} \lambda_{n-k+1}(P^{-1}(t))dt > n$$

where

$$S_\lambda^k = \Big\{ t \in (0,\infty); \quad \lambda_k\Big(\int_0^t Q(s)ds \Big) \geq \lambda \Big\}.$$

The proof of the above theorem will be based on the following notation and lemmas:

On the set S of all symmetric $n \times n$ matrices we define a relation $\underset{k}{\geq}$, $1 \leq k \leq n$ as follows:

DEFINITION 4.5.2: Two symmetric $n \times n$ matrices A and B are $A \underset{k}{\geq} B$ if and only if $0 \underset{k}{\leq} (A - B)$ iff $\lambda_{n-k+1}(A - B) \geq 0$.

Note that in the case $k = 1$, the notion $\underset{k}{\geq}$ is nothing but the notion of positive semi-definiteness for symmetric matrices.

LEMMA 4.5.3. *If* A *and* B *are positive semi-definite such that* $A > 0$ *and* $A^2 \underset{k}{\geq} B^2$, *then* $A \underset{k}{\geq} B$.

LEMMA 4.5.4. *If* C *and* A *are positive definite then for any symmetric matrix* B,

$$ACA \underset{k}{\geq} BCB \quad \textit{implies} \quad A \underset{k}{\geq} B.$$

Negating Lemma 4.5.4, we have

LEMMA 4.5.5. *If C and A are positive definite then for a symmetric matrix B,*

$$A \underset{k}{\gtrless} B \Longrightarrow ACA \underset{k}{\gtrless} BCB.$$

PROOF OF LEMMA 4.5.3. Taking $X = A - B$ note that

$$\begin{aligned} A^2 - B^2 &= A^2 - (A - X)^2 \\ &= AX + XA - X^2. \end{aligned}$$

Hence we get

$$AX + XA = (A^2 - B^2) + X^2.$$

As X^2 is positive and $(A^2 - B^2) \underset{k}{\geq} 0$ we get

$$AX + XA \underset{k}{\geq} 0.$$

That is, there exists a subspace V_{n-k+1} of dimension $n - k + 1$ such that, for u in V_{n-k+1} we have

$$u^*(AX + XA)u \geq 0.$$

Suppose that the lemma is false. This will imply that there exists an orthonormal set $u_1, u_2, \ldots, u_k$ such that

$$u_i^* X u_i = \lambda_{n-i+1}(X) < 0$$

for $1 \leq i \leq k$.

Now for these u_i, $1 \leq i \leq k$, we have

$$\begin{aligned} u_i^*(AX + XA)u_i &= 2u_i^* AX u_i \\ &= 2\lambda_{n-i+1}(X) u_i^* A u_i < 0. \end{aligned}$$

This contradicts the observation made earlier.

PROOF OF LEMMA 4.5.4. We have

$$ACA \underset{k}{\geq} BCB$$

if and only if

$$\lambda_{n-k+1}(ACA - BCB) \geq 0.$$

Let $C^{1/2}$ be a Hermitian square root of C. It follows that

$$\lambda_{n-k+1}(C^{1/2}ACAC^{1/2} - C^{1/2}BCBC^{1/2}) \geq 0.$$

That is,

$$\lambda_{n-k+1}((C^{1/2}AC^{1/2})^2 - (C^{1/2}BC^{1/2})^2) \geq 0.$$

From the definition of $\underset{k}{\geq}$ we have

$$(C^{1/2}AC^{1/2})^2 \underset{k}{\geq} (C^{1/2}BC^{1/2})^2.$$

Using Lemma 4.5.3, we get

$$C^{1/2}AC^{1/2} \underset{k}{\geq} C^{1/2}BC^{1/2}$$

which in turn gives (since $C^{1/2}$ is nonsingular)

$$A \underset{k}{\geq} B.$$

PROOF OF THE THEOREM 4.5.1. We use the Riccati Technique to obtain a contradiction.

If (4.4.1) is not oscillatory we obtain

$$R(t) = R(0) + \int_0^t Q(s)ds + \int_0^t R(s)P^{-1}(s)R(s)ds$$

$$= R(0) + Q_1(t) + \int_0^t R(s)P^{-1}(s)R(s)ds \qquad (4.5.1)$$

where $R(t) = -P(t)Y'(t)Y^{-1}(t)$ is symmetric. Defining

$$F(t) = R(0) + Q_1(t)$$

$$A(t) = \int_0^t R(s)P^{-1}(s)R(s)ds,$$

we note that $A(0) = 0$, $A(t) \geq 0$, $A'(t) \geq 0$. From the definition of $S_\lambda^{(k)}$ and $F(t)$, for $\delta \in (0,1)$, by taking λ sufficiently large we obtain

$$F(t) \underset{k}{\not\leq} \delta\lambda I \quad \text{for} \quad t \in S_\lambda^{(k)}.$$

It follows that

$$F(t) + A(t) \underset{k}{\not\leq} \delta\lambda I + A(t). \qquad (4.5.2)$$

Now

$$A'(t) = R(t)P^{-1}(t)R(t)$$

$$= (F(t) + A(t))P^{-1}(t)(F(t) + A(t))$$

$$\underset{k}{\not\leq} (\delta\lambda I + A(t))(P^{-1}(t))(\delta\lambda I + A(t)).$$

We have

$$(\delta\lambda I + A(t))^{-1} A'(t)(\delta\lambda I + A(t))^{-1} \underset{k}{\leq} P^{-1}(t). \tag{4.5.3}$$

That is

$$0 \underset{k}{\leq} (P^{-1}(t) - W(t)) \tag{4.5.4}$$

where $W(t) = (\delta\lambda I + A(t))^{-1} A'(t)(\delta\lambda I + A(t))^{-1}$. Hence we get

$$\lambda_{n-k+1}\big(P^{-1}(t) - W(t)\big) < 0$$

which implies that

$$\sum_{i=1}^{k} \lambda_{n-i+1}\big(P^{-1}(t) - W(t)\big) < 0.$$

That is, $\rho_k\big(P^{-1}(t) - W(t)\big) < 0$ and hence

$$0 < \sigma_k(W - P^{-1}) \leq \sigma_k(W) + \sigma_k(-P^{-1})$$

implying

$$\sigma_k(W) > -\sigma_k(-P^{-1}) = \rho_k(P^{-1}).$$

Therefore we get

$$\rho_k(P^{-1}) < \sigma_k(W) \leq Tr(W) \quad (\text{as } W \geq 0)$$

$$= Tr\Big(-\frac{d}{dt}(\delta\lambda I + A(t))^{-1}\Big).$$

Integrating through $S_\lambda^{(k)}$ from 0 to T we obtain

$$Tr(\delta\lambda I)^{-1} - Tr\left((\delta\lambda I + A(T))^{-1}\right) > \int_{S_\lambda^{(k)}} \rho_k(P^{-1}(s))ds$$

$$\geq \int_{S_\lambda^{(k)}} \lambda_{n-k+1}(P^{-1}(s))ds.$$

It follows that

$$Tr(\delta\lambda I)^{-1} > \int_{S_\lambda^{(k)}} \lambda_{n-k+1}(P^{-1}(s))ds.$$

That is

$$\frac{n}{\delta\lambda} > \int_{S_\lambda^{(k)}} \lambda_{n-k+1}(P^{-1}(s)ds.$$

Thus we get

$$\lambda \int_{S_\lambda^{(k)}} \lambda_{n-k+1}(P^{-1}(s))ds \leq \frac{n}{\delta} \quad \text{for large} \quad \lambda.$$

By letting $\lambda \to \infty$, we obtain a contradiction. □

We remark that examples may be visualized to realize that Theorem 4.5.1 can be applied even when Theorem 4.4.4 is not applicable.

CHAPTER V

COMPARISON THEOREMS

5.1. Introduction

Consider the second order linear differential scalar equation

$$x'' + kq(t)x = 0. \tag{5.1.1}$$

The function $q(t)$ is an oscillation coefficient if for all $k \in (0, K)$, the solutions to the above equation (5.1.1) are all oscillatory. This notion of oscillation coefficient was introduced in the papers of Utz [53] and Waltman [55].

Clearly the function $q(t) \geq 0$ is an oscillation coefficient if and only if equation (5.1.1) has oscillatory solutions for all $k > 0$.

For the linear second order differential scalar equation

$$(p(t)x')' + q(t)x = 0, \quad t \in [0, \infty) \tag{5.1.2}$$

where $p(t) > 0$ and $p(t)$, $q(t)$ are continuous, as anticipated by Hartman and Wintner [27], Fink and St. Mary [23] have shown that to each $q(t)$ one can associate a number k_0, $0 \leq k_0 \leq +\infty$ such that the equation

$$(p(t)y')' + kq(t)y = 0 \tag{5.1.3}$$

is oscillatory provided the number k satisfies $k > k_0$, and nonoscillatory if $0 \leq k \leq k_0$.

THEOREM 5.1.1 [23]. *If the self adjoint differential equation*

$$(p(t)x')' + kq(t)x = 0, \quad t \in [0, \infty)$$

has zeroes at a *and* b, $a < b$, *and* $j > k > 0$, *then*

$$(p(t)x')' + jq(t)x = 0$$

has at least one zero in $[a, b]$.

□

This is in the form of the Sturm comparison theorem, however $jq(t) \not\geq kq(t)$ unless $q(t) \geq 0$. In fact, $jq(t) \geq kq(t)$ on $\{t/q(t) \geq 0\}$ and $jq(t) < kq(t)$ on $\{t/q(t) < 0\}$. It is not true that $q_1 \geq q$ on $\{t/q(t) \geq 0\}$; $q_1 \leq q$ on $\{t/q(t) < 0\}$ implies

$$(p(t)x')' + q_1(t)x = 0 \tag{5.1.4}$$

oscillates faster than

$$(px')' + qx = 0. \tag{5.1.5}$$

By taking $q_1(t) = q(t)$ on $\{t/q(t) \geq 0\}$ and $q_1(t) = 2q(t)$ on $\{t/q(t) < 0\}$ and by using the Sturm comparison theorem, one observes that equation (5.1.4) oscillates slower than equation (5.1.5) and hence it can be concluded that the above theorem is not valid for a non-constant multiplier of a given function, in general. However, the result (Theorem 5.5.1) also follows easily from Sturm's First Comparison Theorem (cf. [26], p.334) by rewriting the equation and observing that $\frac{1}{j}p(t) < \frac{1}{k}p(t)$ if $j > k > 0$.

But Erbe [16] showed that the constant k can be replaced by a class of functions $k(t)$. That is, multiplying the coefficient $q(t)$ by a certain class of nonconstant functions also preserves the oscillation property.

THEOREM 5.1.2 [16]. *Let $x(t)$ be a nontrivial solution of*

$$(p(t)x')' + q(t)x = 0$$

satisfying $x(a) = x(b) = 0$. Let $k(t) \in C^{(1)}[a,b]$ satisfying $k(t) \geq 1$ on $[a,b]$ and assume pk' is nonincreasing on $[a,b]$. Then every solution of

$$(p(t)x')' + k(t)q(t)x = 0$$

has a zero on (a,b).

□

In [33] Kwong showed that Erbe's result is still true for a much wider class of functions.

THEOREM 5.1.3. [33]. *Let $k(t) \in C^{(1)}[0,\infty)$ such that*

(i) $k(t) \geq 1$

(ii) $2p(t)k'(t) - 3\int_0^t \frac{p(s)k'^2(s)}{k(s)}ds$ *is nonincreasing for large t. If the equation*

$$(px')' + qx = 0$$

is oscillatory, then so is

$$(px')' + k(t)q(t)x = 0.$$

5.2. Comparison problems for systems

Consider the second order $n \times n$ matrix differential equation

$$(P(t)Y')' + Q(t)Y(t) = 0, \quad t \in [a, \infty) \tag{5.2.1}$$

where P, Q, Y are $n \times n$ real continuous matrix functions with $P(t), Q(t)$ symmetric and $P(t)$ positive definite for $t \in [a, \infty)$. We also have the associated vector system

$$(P(t)y')' + Q(t)y = 0 \tag{5.2.2}$$

where $y = \text{col}(y_1, y_2, \ldots, y_n)$ is an n-vector.

Our interest now is the following:

Assume that the equation (5.2.1) is oscillatory. What are the relations between P and P_1, Q and Q_1 which ensure that the equation

$$(P_1Y')' + Q_1Y = 0, \quad t \in [a, \infty)$$

is also oscillatory?

In one direction, as we have seen, by imposing a partial ordering on the set of Hermitian matrices, several authors have proved a number of results using a direct relation between P and P_1, Q and Q_1 in terms of the partial ordering. The other direction is the problem of determining for what matrices $A(t)$ for which the second order $n \times n$ matrix differential systems

$$(PY')' + AQY = 0$$

and

$$(PY')' + AQAY = 0$$

are oscillatory? It is this latter problem we shall concentrate on in this chapter.

5.3. Functionally commutative coefficients

First, we recall the definition of functional commutativity of a matrix.

DEFINITION 5.3.1: A continuous $n \times n$ matrix valued function $Q(t)$ is said to be functionally commutative on an interval J if

$$Q(s)Q(t) = Q(t)Q(s) \quad \text{for all} \quad s,t \in J.$$

We need the following characterization of functionally commutative matrices Freedman [22].

LEMMA 5.3.2. *Let $Q(t)$ be a diagonalizable $n \times n$ matrix which is functionally commutative on an interval J. Then $Q(t)$ has a constant set of eigenvectors and therefore there exists a constant matrix P such that $Q(t) = P^{-1}D(t)P$, where $D(t)$ is a diagonal matrix.*

LEMMA 5.3.3. *Let $A_1, \ldots, A_N$ be a given set of $n \times n$ matrices. Then the following statements are equivalent:*

(i) A_i is diagonalizable for each i, and A_i commute pairwise;

(ii) $A_1, \ldots, A_N$ can be simultaneously diagonalized.

The above lemma 5.3.3 may be found in Drazin, Dungey and Greuenberg [14]. For a differential equation with functionally commutative coefficients, we have

THEOREM 5.3.4. *Suppose that*

$$Y'' + Q(t)Y = 0, \quad t \in [a, \infty) \tag{5.3.1}$$

is oscillatory and assume that

(i) $Q(t)$ *is functionally commutative, symmetric and continuous;*

(ii) $A(t)$ *and* $Q(t)$ *commute;*

(iii) $A(t)$ *is symmetric, twice differentiable such that* $A(t) \geq I_n$ *and* $A''(t) \leq 0$.

Then

$$Y'' + A(t)Q(t)Y = 0 \tag{5.3.2}$$

is oscillatory.

PROOF: Since $Q(t)$ is symmetric and functionally commutative on the interval $J = [a, \infty)$, we can use the above lemma [5.3.2] and write

$$Q(t) = P^{-1}D(t)P \tag{5.3.3}$$

where $D(t)$ is diagonal and P is unitary. Using $X(t) = PY(t)$ in equation (5.3.1) we obtain

$$X''(t) + D(t)X(t) = 0. \tag{5.3.4}$$

The above equation (5.3.4) is oscillatory. Actually, $Y(t)$ is a prepared solution of (5.3.1) if and only if $X(t)$ is a prepared solution of (5.3.4), and in fact both equations exhibit the same oscillatory or nonoscillatory behavior. Thus, in particular the soltuion $X_0(t)$ satisfying

$$X_0(a) = 0, \quad X_0'(a) = I \tag{5.3.5}$$

is an oscillatory solution of (5.3.4). The solution of (5.3.4) satisfying the initial condition (5.3.5) is diagonal for all $t \geq a$, and so $X_0(t) = \text{diag}\{x_1(t), x_2(t), \ldots, x_n(t)\}$. For this solution, (5.3.4) can be written in the form

$$x_i'' + d_i(t)x_i(t) = 0, \quad i = 1, 2, \ldots, n$$

where $D(t) = \text{diag}\{d_1(t), d_2(t), \ldots, d_n(t)\}$.

Since $\det X_0(t) = \prod_{i=1}^{n} x_i(t)$, $X_0(t)$ oscillates if and only if one of the functions $x_i(t)$ oscillates. That is, the equation

$$x_{i_0}'' + q_{i_0}(t)x_{i_0}(t) = 0 \tag{5.3.6}$$

is oscillatory for some i_0, $1 \leq i_0 \leq n$. Therefore by Theorem 5.1.3, we get

$$x_{i_0}'' + \lambda_{i_0}(t)q_{i_0}x_{i_0}(t) = 0 \tag{5.3.7}$$

is oscillatory for some i_0 and for every $\lambda_{i_0}(t)$ such that $\lambda_{i_0}(t) \geq 1$ and $\lambda_{i_0}''(t) \leq 0$. Since $Q(t)$ and $A(t)$ are diagonalizable and commute, Lemma 5.3.3 shows that $Q(t)$ and $A(t)$ can be diagonalized by the same constant matrix P. That is

$$A(t)Q(t) = P^{-1}\Lambda DP \tag{5.3.8}$$

where $\Lambda = P^{-1}AP$ and $\Lambda(t) = \text{diag}\{\lambda_1(t), \ldots, \lambda_n(t)\}$. Now taking $X(t) = PY(t)$, where $Y(t)$ is a solution of 5.3.2, we get

$$X'' + \Lambda(t)D(t)X = 0. \tag{5.3.9}$$

Now using the conditions on $A(t)$ and observation (5.3.7) we conclude that equation (5.3.9) is oscillatory. Hence $Y(t)$ is an oscillatory solution of equation (5.3.2). □

In the case of a constant matrix A we have

COROLLARY 5.3.5. *Suppose equation (5.3.1) is oscillatory and assume that*

(i) Q *is functionally commutative;*

(ii) A *and* Q *commute;*

(iii) $A = A^* \geq I_n$;

then (5.3.2) is oscillatory.

5.4. Comparison results

In the following we no longer require functional commutativity for the coefficient matrix $Q(t)$. Using the Variational Technique, for vector systems, we have

THEOREM 5.4.1. *Suppose*

$$y'' + Q(t)y = 0 \tag{5.4.1}$$

is oscillatory. Then

$$y'' + AQ(t)Ay = 0 \tag{5.4.2}$$

is oscillatory for all constant matrices $A = A^* \geq I_n$.

PROOF: Since A is symmetric and $A \geq I$, there exists an $n \times n$ real orthogonal matrix P such that

$$P^{-1}(A - I)P = \Gamma \geq 0 \tag{5.4.3}$$

where Γ is a diagonal matrix. Hence we have

$$P^{-1}AP = I + \Gamma \tag{5.4.4}$$

which says that P also diagonalizes A. From the conditions on A it follows that the eigenvalues of A^2 are greater than or equal to 1, which in turn implies that the eigenvalues of A^{-2} are less than are equal to 1.

From Theorem [2.5.1], for some interval $[a, b]$ and for some u in $A_1(a, b)$ we have

$$J_Q(u) = \int_a^b [|u'|^2 - u^*Qu]\,dt < 0. \tag{5.4.5}$$

Taking $v = A^{-1}u$ we have, v is in the same space as u, and

$$\begin{aligned} |v'|^2 &= (v' \cdot v') \\ &= v'^* v' \\ &= u'^* A^{-2} u'. \end{aligned} \tag{5.4.6}$$

We also have

$$\begin{aligned} v^*AQAv &= u^*A^{-1}AQAA^{-1}u \\ &= u^*Qu. \end{aligned} \tag{5.4.7}$$

Using the above observations we get

$$\begin{aligned} J_{AQA}(v) &= \int_a^b [|v'|^2 - v^*AQAv]\,dt \\ &= \int_a^b [u'^*A^{-2}u' - u^*Qu]\,dt \\ &\leq \int_a^b [|u'|^2 - u^*Qu]\,dt < 0 \end{aligned} \tag{5.4.8}$$

and so by Theorem [2.5.1], the result follows.

A more general result is the following

THEOREM 5.4.2. *Let*

$$(P(t)y')' + Q(t)y = 0 \tag{5.4.9}$$

be oscillatory. Then

$$(P(t)y')' + AQAy = 0 \tag{5.4.10}$$

is oscillatory for all constant matrices A for which

(i) $A = A^* \geq I_n$;

(ii) A *and* $P(t)$ *commute.*

□

For matrix systems we have the following result for nonconstant matrix multiplicative coefficients.

THEOREM 5.4.3. *Suppose*

$$(PY')' + QY = 0 \tag{5.4.11}$$

is oscillatory. Assume that B is symmetric and satisfies

(i) $BB' = B'B,\ B \geq I$;

(ii) $P^{\frac{1}{2}}B = BP^{\frac{1}{2}}$;

(iii) $PB' = B'P$;

(iv) $(PH')'H \geq 0$ *where* $H = B^{-1}$.

Then

$$(PZ')' + BQBZ = 0 \tag{5.4.12}$$

is also oscillatory.

PROOF: If equation (5.4.12) has a nontrivial prepared solution $Z = Z(t)$ with $\det Z(t) \neq 0$ for $t \geq t_0$ then taking $S(t) = -PZ'Z^{-1}$ we have

$$S'(t) = -(PZ')'Z^{-1} + PZ'Z^{-1}Z'Z^{-1}$$

$$= BQB + S(t)P^{-1}S(t).$$

Therefore $S(t)$ is a symmetric solution of the Riccati integral equation

$$S(t) = S(t_0) + \int_{t_0}^{t} BQBds + \int_{t_0}^{t} S(s)P^{-1}(s)S(s)ds. \tag{5.4.13}$$

By taking $R(t) = B^{-1}S(t)B^{-1}$ we note that

$$R'(t) = -B^{-1}B'B^{-1}S(t)B^{-1} + B^{-1}S'(t)B^{-1} - B^{-1}S(t)B^{-1}B'B^{-1}$$

$$= Q + B^{-1}S(t)P^{-1}S(t)B^{-1} - B^{-1}B'B^{-1}S(t)B^{-1} - B^{-1}S(t)B^{-1}B^{1}B^{-1}$$

$$= Q + B^{-1}S(t)B^{-1}BP^{-1}BB^{-1}S(t)B^{-1} - B^{-1}B'R - RB'B^{-1}$$

$$= Q + RBP^{-1}BR - (RB'B^{-1} + B^{-1}B'R). \tag{5.4.14}$$

Integrating, we obtain

$$R(t) = R(c) + \int_{c}^{t} Q(s)ds + \int_{c}^{t} (RBP^{-1}BR - B^{-1}B'R - RB'B^{-1})ds$$

$$= R(c) + \int_{c}^{t} Q(s)ds + \int_{c}^{t} (RBP^{-1}BR - B^{-1}B'B^{-1}BR - RBB^{-1}B'B^{-1})ds.$$

That is,

$$R(t) = R(c) + \int_{c}^{t} Q(s)ds + \int_{c}^{t} (RBP^{-1}BR + (B^{-1})'BR + RB(B^{-1})')ds.$$

Now by taking $W(t) = R + PHH'$, where $H = B^{-1}$, we observe that

$$W(t)H^{-1}P^{-1}H^{-1}W(t) = (R + PHH')H^{-1}P^{-1}H^{-1}(R + PHH')$$

$$= RH^{-1}P^{-1}H^{-1}R + RH^{-1}P^{-1}H^{-1}PHH'$$

$$+ PHH'H^{-1}P^{-1}H^{-1}R + PHH'H^{-1}P^{-1}H^{-1}PHH'.$$

Using $PB = BP$ and $HH' = H'H$ we get

$$W(t)H^{-1}P^{-1}H^{-1}W(t) = RBP^{-1}BR + RH^{-1}H' + PH'P^{-1}H^{-1}R + P(H')^2.$$

Using $PB' = B'P$ we get

$$W(t)H^{-1}P^{-1}H^{-1}W(t) = RBP^{-1}BR + RH^{-1}H' + H'H^{-1}R + P(H')^2.$$

Using this in the above Riccati equation, we get

$$W(t) = PHH' - \int_c^t P(H')^2 + R(c) + \int_c^t Q(s)ds + \int_c^t W(s)H^{-1}P^{-1}H^{-1}W(s)ds$$

$$= \tilde{Q}(t) + R(c) + \int_c^t Q(s)ds + \int_c^t W(s)BP^{-1}BW(s)ds. \qquad (5.4.15)$$

We note that (5.4.15) is the Riccati equation corresponding to the matrix differential equation

$$(B^{-2}PZ')' + (\tilde{Q}' + Q)Z = 0. \qquad (5.4.16)$$

Since Riccati equation (5.4.15) has a solution we get that equation (5.4.16) is nonoscillatory. Since P has symmetric square root which commutes with B and $B \geq I$ we get $B^{-2}P \leq P$. Since $(PH')'H \geq 0$, we get

$$(PHH')' \geq P(H')^2.$$

That is $\tilde{Q}' \geq 0$. Hence we get equation (5.4.11) is nonoscillatory. This contradiction implies that equation (5.4.12) is oscillatory.

5.5. Disconjugacy

Consider the second order vector differential equation

$$(P(t)y')' + Q(t)y = 0. \tag{5.5.1}$$

In addition to the usual assumptions on $P(t)$ and $Q(t)$ we assume that $P(t)$ is functionally commutative on the interval J. Then there exists a constant matrix T such that $TT^* = I$ and $P(t) = T^{-1}D(t)T$ where $D(t) = \text{ diag}\big(d_1(t), d_2(t), \ldots, d_n(t)\big)$, $d_i(t) > 0$. Taking $x(t) = Ty(t)$ we get that $x(t)$ satisfies the equation

$$(D(t)x')' + Q_1(t)x = 0 \tag{5.5.2}$$

where

$$Q_1(t) = TQ(t)T^{-1}. \tag{5.5.3}$$

Let $x(t) = col(x_1, x_2, \ldots, x_n)$ be a solution of (5.5.2). Taking $z(t) = col(\epsilon x, Dx')$ for some $\epsilon > 0$, we get

$$z' = \begin{bmatrix} \epsilon x' \\ -Q_1 x \end{bmatrix} = \begin{bmatrix} 0 & \epsilon D^{-1} \\ -\epsilon^{-1}Q_1 & 0 \end{bmatrix} \begin{bmatrix} \epsilon x \\ Dx' \end{bmatrix}.$$

That is

$$z' = C(t, \epsilon)z \tag{5.5.4}$$

where

$$C(t,\epsilon) = \begin{bmatrix} 0 & \epsilon D^{-1} \\ -\epsilon^{-1}Q_1 & 0 \end{bmatrix}. \tag{5.5.5}$$

Using a result of Nehari [47], we get that equation (5.5.4) is nonoscillatory on $[a,b]$ if

$$\int_a^b \|C\|dt < \frac{\pi}{2} \tag{5.5.6}$$

where $\|C\|$ denotes the matrix norm $\sup_{\|v\|=1} \|Bv\|$ and $\|v\|$ is the Euclidean norm of vector v.

Taking $u = col(z_1, z_2, \ldots, z_n)$ and $v = col(z_{n+1}, \ldots, z_{2n})$ we note that

$$\begin{aligned}
\|Cz\|^2 &= (Cz, Cz) \\
&= \left(\begin{bmatrix} \epsilon D^{-1}v \\ -\epsilon^{-1}Q_1 u \end{bmatrix}, \begin{bmatrix} \epsilon D^{-1}v \\ -\epsilon^{-1}Q_1 u \end{bmatrix} \right) \\
&= \epsilon^2 \|D^{-1}v\|^2 + \epsilon^{-2}\|Q_1 u\|^2 \\
&\leq \epsilon^2 \|D^{-1}\|^2 \|v\|^2 + \epsilon^{-2}\|Q_1\|^2 \|u\|^2 \\
&\leq \max\left(\epsilon^2\|D^{-1}\|^2,\ \epsilon^{-2}\|Q_1\|^2\right)\|z\|^2
\end{aligned}$$

hence

$$\|C\| \leq \max\left(\epsilon\|D^{-1}\|,\ \epsilon^{-1}\|Q_1\|\right). \tag{5.5.7}$$

If $x = (x_1, \ldots, x_n)^T$ is a solution of (5.5.2) then taking $w = \begin{pmatrix} x \\ Dx' \end{pmatrix}$ we can reduce the second order equation (5.5.2) to the first-order system

$$w' + C_1 w = 0 \tag{5.5.8}$$

where

$$C_1 = \begin{bmatrix} 0 & -D^{-1} \\ -Q & 0 \end{bmatrix}.$$

If $x = (x_1, \ldots, x_n)$ is a nontrivial solution such that $x(a) = x(b) = 0$ then $x_i(a) = x_i(b) = 0$ for $i = 1, 2, \ldots, n$. By Rolle's theorem we get $y_i'(t_i) = 0$ for some t_i, $a < t_i < b$, $i = 1, \ldots, n$. This means that every component of w has a zero and hence equation (5.5.8) is oscillatory. Thus nonoscillation of (5.5.4) implies that equation (5.5.2) is right and left disfocal and hence

THEOREM 5.5.1. *If*

$$\int_a^b \max\left(\epsilon\|D^{-1}\|,\ \epsilon^{-1}\|Q_1\|\right)dt < \frac{\pi}{2}$$

for some $\epsilon > 0$, then equation (5.5.4) is nonoscillatory on the interval $I = [a, b]$.

□

THEOREM 5.5.2. *If for some $\epsilon > 0$,*

$$\int_a^b \max\left(\epsilon\|D^{-1}\|,\ \epsilon^{-1}\|Q_1\|\right)dt < \frac{\pi}{2}$$

then equation (5.5.1) is disconjugate and right - and left - disfocal on $[a, b]$.

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